Stochastic Nonlinear Control: A Unified Framework for Stability, Dissipativity, and Optimality

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नासदासीन्नो सदासीत्तदानीं नासीद्रजो नो व्योमा परो यत् |
किमावरीवः कुह कस्य शर्मन्नम्भः किमासीद्गहनं गभीरम् ॥ १ ॥

Then even nothingness was not, nor existence,
There was no air then, nor the heavens beyond it.
What covered it? Where was it? In whose keeping?
Was there then cosmic water, in depths unfathomed?

न मृत्युरासीदमृतं न तर्हि न रात्र्या अह्न आसीत्प्रकेतः |
आनीदवातं स्वधया तदेकं तस्माद्धान्यन्न परः किञ्चनास ॥२ ॥

Then there was neither death nor immortality
nor was there then the torch of night and day.
The One breathed windlessly and self-sustaining.
There was that One then, and there was no other.

तम आसीत्तमसा गूहळमग्रे प्रकेतं सलिलं सर्वाऽइदम् |
तुच्छ्येनाभ्वपिहितं यदासीत्तपसस्तन्महिनाजायतैकम् ॥३ ॥

At first there was only darkness wrapped in darkness.
All this was only unillumined water.
That One which came to be, enclosed in nothing,
arose at last, born of the power of heat.

कामस्तदग्रे समवर्तताधि मनसो रेतः प्रथमं यदासीत् |
सतो बन्धुमसति निरविन्दन्हृदि प्रतीष्या कवयो मनीषा ॥४ ॥

In the beginning desire descended on it -
that was the primal seed, born of the mind.
The sages who have searched their hearts with wisdom
know that which is kin to that which is not.

तिरश्चीनो विततो रश्मिरेषामधः स्विदासीदुपरि स्विदासीत् |
रेतोधा आसन्महिमान आसन्त्स्वधा अवस्तात्प्रयतिः परस्तात् ॥५ ॥

And they have stretched their cord across the void,
and know what was above, and what below.
Seminal powers made fertile mighty forces.
Below was strength, and over it was impulse.

को अद्धा वेद क इह प्र वोचत्कुत आजाता कुत इयं विसृष्टिः |
अर्वाग्देवा अस्य वसिर्जनेनाथा को वेद यत आबभूव ॥६ ॥

But, after all, who knows, and who can say
Whence it all came, and how creation happened?
the gods themselves are later than creation,
so who knows truly whence it has arisen?

इयं विसृष्टिर्यत आबभूव यदि वा दधे यदि वा न |
यो अस्याध्यक्षः परमे व्योमन्त्सो अङ्ग वेद यदिवा न वेद ॥७ ॥

Whence all creation had its origin,
he, whether he fashioned it or whether he did not,
he, who surveys it all from highest heaven,
he knows - or maybe even he does not know.

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Summary

In this dissertation, we present a unified framework for stability, dissipativity, and optimality for stochastic nonlinear control systems. First, we develop a complete theory for stochastic semistability. Semistability is the property whereby the solutions of a stochastic dynamical system almost surely converge to (not necessarily isolated) Lyapunov stable in probability equilibrium points determined by the system initial conditions. Specifically, we developed Lyapunov and converse Lyapunov theorems for stochastic semistable nonlinear dynamical systems. In particular, we provide necessary and sufficient Lyapunov conditions for stochastic semistability and show that stochastic semistability implies the existence of a continuous Lyapunov function whose infinitesimal generator decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

Next, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for nonlinear stochastic dynamical systems. Specifically, we provide a simplified and tutorial framework for stochastic optimal control and focus on connections between stochastic Lyapunov theory and stochastic Hamilton-Jacobi-Bellman theory. In particular, we show that asymptotic stability in probability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function which can clearly be seen to be the solution to the steady-state form of the stochastic Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both stochastic stability and optimality. In addition, we develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the stochastic stabilization problem. These results are then used to provide exten-

sions of the nonlinear feedback controllers obtained in the literature that minimize general polynomial and multilinear performance criteria.

Using the aforementioned optimal nonlinear analysis and feedback control framework, we also develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for partial stability and partial-state stabilization of stochastic dynamical systems. Partial asymptotic stability in probability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state which can clearly be seen to be the solution to the steady-state form of the stochastic Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both partial stability in probability and optimality. The overall framework provides the foundation for extending optimal linear-quadratic stochastic controller synthesis to nonlinear-nonquadratic optimal partial-state stochastic stabilization. Connections to optimal linear and nonlinear regulation for linear and nonlinear time-varying stochastic systems with quadratic and nonlinear-nonquadratic cost functionals are also provided. We also develop optimal feedback controllers for affine stochastic nonlinear systems using an inverse optimality framework tailored to the partial-state stochastic stabilization problem and use this result to address polynomial and multilinear forms in the performance criterion.

In many practical applications, stability with respect to part of the system's states is often necessary with finite-time convergence to the equilibrium state of interest. Finite-time partial stability involves dynamical systems whose part of the trajectory converges to an equilibrium state in finite time. Using our proposed analysis and control synthesis framework, we address finite-time partial stability in probability and uniform finite-time partial stability in probability for nonlinear stochastic dynamical systems. Specifically, we provide Lyapunov conditions involving a Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers for guaranteeing finite-time partial stability in probability. In addition, we show that finite-time partial stability in probability leads to uniqueness of solutions in forward time and we establish necessary and sufficient conditions for almost sure continuity of the settling-time operator of the nonlinear stochastic dynamical system. Next, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control design for finite-time partial stochastic stability and finite-time, partial-state stochastic stabilization. Finite-time partial stability in probability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state and can clearly be seen to be the solution to the steady-state form of the stochastic Hamilton-Jacobi-Bellman equation guaranteeing both finite-time, partial-state stability and optimality.

Building on our optimal control framework, we extend our results to optimal and inverse optimal stochastic differential games. Specifically, we consider a two-player stochastic differential game problem over an infinite time horizon where the players invoke controller and stopper strategies on a nonlinear stochastic differential game problem driven by Brownian motion. The optimal strategies for the two players are given explicitly by exploiting connections between stochastic Lyapunov stability theory and stochastic Hamilton-Jacobi-Isaacs theory. In particular, we show that asymptotic stability in probability of the differential game problem is guaranteed by means of a Lyapunov function which can clearly be seen to be the solution to the steady-state form of the stochastic Hamilton-Jacobi-Isaacs equation, and hence, guaranteeing both stochastic stability and optimality of the closed-loop control and stopper policies. In addition, we develop optimal feedback controller and stopper policies for affine nonlinear systems using an inverse optimality framework tailored to the stochastic differential game problem. These results are then used to provide extensions of the linear feedback controller and stopper policies obtained in the literature to nonlinear feedback controllers and stoppers that minimize and maximize general polynomial and multilinear performance criteria.

Finally, we develop stochastic dissipativity theory for nonlinear dynamical systems using basic input-output and state properties. Specifically, a stochastic version of dissipativity using both an input-output as well as a state dissipation inequality in expectation for controlled Markov diffusion processes is presented. The results are then used to derive extended Kalman–Yakubovich–Popov conditions for characterizing necessary and sufficient conditions for stochastic dissipativity of stochastic dynamical systems using two-times continuously differentiable storage functions. In addition, feedback interconnection stability in probability results for stochastic dynamical systems are developed thereby providing a generalization of the small gain and positivity theorems to stochastic systems.

Chapter 1

Introduction

1.1. Motivation and Goals

In the first part of this dissertation, we develop new and novel results for semistability of stochastic dynamical systems. Semistability is the property of a dynamical system whereby its trajectories converge to (not necessarily isolated) Lyapunov stable equilibria. Semistability, rather than asymptotic stability, is the appropriate notion of stability for systems having a continuum of equilibria. Examples of such systems arise in chemical kinetics [27], adaptive control [18], compartmental modeling [46], thermodynamics [47] and, more recently, collaborative control of a network of autonomous agents [53, 54]. In all these examples, the system trajectories converge to limit points that depend continuously on the system initial conditions.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point as examples in [18] show. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

For linear deterministic systems, semistability was originally defined in [25] and applied to matrix second-order systems in [15]. References [18] and [20] extended the notion of semistability to nonlinear deterministic systems and give Lyapunov results for semistability. Semistability was also addressed in [53, 54] for consensus protocols in nonlinear dynamical networks, with [54] giving new Lyapunov theorems as well as the first converse Lyapunov theorem for semistability which holds with a smooth (i.e., infinitely differentiable) Lyapunov function. Extensions of semistability to stochastic dynamical systems has not been addressed in the literature.

Under certain conditions nonlinear controllers offer significant advantages over linear controllers. In particular, if the plant dynamics and/or system measurements are nonlinear [12, 103], the plant/measurement disturbances are either nonadditive or non-Gaussian, the performance measure considered in nonquadratic [11, 87, 93, 97, 100], the plant model is uncertain [9, 70, 85], or the control signals/state amplitudes are constrained [21, 91], then nonlinear controllers yield better performance than the best linear controllers.

In [14] the current status of continuous-time, nonlinear-nonquadratic optimal control problems for *deterministic* dynamical systems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [14] are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both *asymptotic* stability and optimality [14,45]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems. In this dissertation, we build on the results of [14, 15] to develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for *stochastic* dynamical systems.

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. Most of the existing control techniques for deterministic dynamical systems in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval.

In order to achieve convergence in finite time for deterministic dynamical systems, the closed-loop system dynamics need to be non-Lipschitzian giving rise to non-uniqueness of solutions in backward time. Uniqueness of solutions in forward time, however, can be preserved in the case of finite-time convergence. Sufficient conditions for deterministic dynamical systems that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [1, 37, 65, 117], whereas [105, 114] give sufficient conditions that ensure uniqueness of solutions for stochastic dynamical systems in forward time in the absence of a uniform Lipschitz continuity and a growth restriction condition on the system drift and diffusion functions. In addition, it is shown in [105, 114] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are sample continuous (i.e., almost surely continuous) functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

Finite-time convergence to a Lyapunov stable equilibrium, that is, *finite-time stability*, was first addressed by Roxin [89] and rigorously studied in [17, 19] for time-invariant deterministic systems using continuous Lyapunov functions. Extensions of finite-time stability to time-varying nonlinear dynamical systems are presented in [49, 81], whereas extensions of finite-time stability to stochastic dynamical systems are reported in [29, 116]. Another important stability notion in many engineering applications is *partial stability*, that is, stability with respect to part of the system's states. In particular, partial stability arises in the study of electromagnetics [120], inertial navigation systems [98], spacecraft stabilization via gimballed gyroscopes and/or flywheels [102], combustion systems [7], vibrations in rotating machinery [72], and biocenology [88], to cite but a few examples. As noted above, the need to consider partial stability in the aforementioned systems arises from the fact that stability notions involve equilibrium coordinates as well as a hyperplane of coordinates that is closed but *not* compact. Hence, partial stability involves motion lying in a subspace instead of an equilibrium point. Extensions of partial stability in probability to stochastic dynamical systems are addressed in [55, 71, 96].

For *deterministic* dynamical systems, *finite-time stabilization*, that is, the problem of finding state-feedback control laws that guarantee finite-time stability of the closed-loop system, as well as the problem of *partial-state stabilization*, wherein stabilization with respect to a subset of the system state variables is desired has been considered in the literature. In particular, finite-time stabilization of second-order systems was considered in [16, 50]. More recently, researchers have considered finite-time stabilization of higher-order systems [51] as well as finite-time stabilization using output feedback [52]. Design of globally strongly stabilizing continuous controllers for linear and nonlinear systems using the theory of homogeneous systems was studied in [19, 86].

Optimal control for finite-time stabilization is addressed in [48], whereas the universal controller given by Sontag [99] is extended in [80] to design a feedback controller for finite-time stabilization. Alternatively, discontinuous finite-time stabilizing feedback controllers have also been developed in the literature [44, 90, 92]. However, for practical implementations, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics. Finally, the problems of partial-state stabilization as well as the combined finite-time, partial-state stabilization problem for deterministic dynamical systems have also been addressed in the literature [69, 72, 102]. The problems of finite-time *stochastic* stabilization, optimal finite-time stochastic stabilization, *optimal* partial-state stochastic stabilization, as well as the combined probelm of *optimal finite-time*, *partial-state stochastic stabilization* have not been addressed in the literature.

A closely related problem to optimal control is the optimal differential game problem. Differential games have been studied in various contexts in the literature including risk-sensitive control [38], mathematical finance [23, 26], communication networks [10], and network resource allocation [61]. The pioneering work on the subject involved a deterministic two-player, zero-sum differential game problem whose solution is characterized by the Hamilton-Jacobi-Isaacs equation [36, 43, 56]. Building on this work, [40] were the first to extend the two-player, zero-sum differential game problem to a stochastic setting and prove that the lower and the upper value functions of this game satisfy the dynamic programming principle. Specifically, they showed that the lower and the upper value functions of this game are the unique viscosity solutions of the associated stochastic Hamilton-Jacobi-Isaacs equation. Furthermore, they showed that these solutions coincide under the Isaacs minimax condition. In [39], the authors extend the results of [40] by relaxing the minimax Isaacs condition and considering a saddle point property that generates approximately optimal control strategies for the maximizing and minimizing players. In particular, even though both players choose specific strategies, in the upper game characterized by the upper value function the strategies chosen by the minimizer are restricted to a subclass of Elliott-Kalton strategies [40].

Many physical and engineering systems are *open systems*, that is, the system behavior is described by an evolution law that involves the system state and the system input with, possibly, an output equation wherein past trajectories together with the knowledge of any inputs define future trajectories (uniquely or nonuniquely) and the system output depends on the instantaneous (present) values of the system state. Dissipativity theory is a system-theoretic concept that provides a powerful framework for the analysis and control design of open dynamical systems based on generalized system energy considerations. In particular, dissipativity theory exploits the notion that numerous physical dynamical systems have certain input-output and state properties related to conservation, dissipation, and transport of mass and energy.

Such conservation laws are prevalent in dynamical systems, in general, and feedback control systems, in particular. The dissipation hypothesis on dynamical systems results in a fundamental constraint on the system dynamical behavior, wherein the stored energy of a dissipative dynamical system is at most equal to sum of the initial energy stored in the system and the total externally supplied energy to the system. Thus, the energy that can be extracted from the system through its input-output ports is less than or equal to the initial energy stored in the system, and hence, there can be no internal creation of energy; only conservation or dissipation of energy is possible.

The key foundation in developing dissipativity theory for *deterministic* nonlinear dynamical systems with continuously differentiable flows was presented by Willems [107, 108] in his seminal two-part paper on dissipative dynamical systems. In particular, Willems [107] introduced the definition of dissipativity for general nonlinear dynamical systems in terms of a *dissipation inequality* involving a generalized system power input, or *supply rate*, and a generalized energy function, or *storage function*. The dissipation inequality implies that the increase in generalized system energy over a given time interval cannot exceed the generalized energy supply delivered to the system during this time interval. The set of all possible system storage functions is convex and every system storage function is bounded from below by the *available system storage* and bounded from above by the *required energy supply*.

In light of the fact that energy notions involving conservation, dissipation, and transport also arise naturally for dissipative diffusion processes, it seems natural that dissipativity theory can play a key role in the analysis and control design of stochastic dynamical systems. Specifically, as in the analysis of deterministic dynamical systems, dissipativity theory for stochastic dynamical systems can involve conditions on drift and diffusion system parameters that render an input, state, and output system dissipative. In addition, robust stability for stochastic dynamical systems with stochastic uncertainty can be analyzed by viewing the uncertain stochastic dynamical system as an interconnection of stochastic dissipative dynamical subsystems. Alternatively, stochastic dissipativity theory can be used to design feedback controllers that add dissipation and guarantee stability robustness in probability allowing stochastic stabilization to be understood in physical terms. As for deterministic dynamical systems [45], stochastic dissipativity theory can play a fundamental role in addressing stochastic robustness [112], risk-sensitive disturbance rejection [77], stability in probability of feedback interconnections, and optimality with averaged performance measurers for stochastic dynamical systems.

Even though several notions of stochastic dissipativity have been considered in the literature [22, 101, 112, 118], a general theory of stochastic dissipativity and stochastic losslessness involving connections between input-output and state properties, which include the notable special cases of stochastic passivity and stochastic finite-gain nonexpansivity using extended Kalman-Yakubovich-Popov conditions in terms of the drift and diffusion terms in the system dynamics, and stability in probability of general feedback interconnections has not been addressed.

1.2. Brief Outline of the Dissertation

In this dissertation, we develop stochastic extensions for each of the aforementioned topics. Specifically, in Chapter 2, we develop Lyapunov and converse Lyapunov theorems for stochastic semistable nonlinear dynamical systems. In Chapter 3, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for nonlinear stochastic dynamical systems. Specifically, we provide a simplified and tutorial framework for stochastic optimal control and focus on connections between stochastic Lyapunov theory and stochastic Hamilton-Jacobi-Bellman theory. In Chapter 4, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for partial stability and partial-state stabilization of stochastic dynamical systems. Partial asymptotic stability in probability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state which can clearly be seen to be the solution to the steady-state form of the stochastic Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both partial stability in probability and optimality.

In Chapter 5, we address finite-time partial stability in probability and uniform finite-time partial stability in probability for nonlinear stochastic dynamical systems. Specifically, we provide Lyapunov conditions involving a Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers for guaranteeing finite-time partial stability in probability. In Chapter 6, we consider a two-player stochastic differential game problem over an infinite time horizon where the players invoke controller and stopper strategies on a nonlinear stochastic differential game problem driven by Brownian motion. The optimal strategies for the two players are given explicitly by exploiting connections between stochastic Lyapunov stability theory and stochastic Hamilton-Jacobi-Isaacs theory. In Chapter 7, we develop stochastic dissipativity theory for nonlinear dynamical systems using basic input-output and state properties. Specifically, a stochastic version of dissipativity using both an input-output as well as a state dissipation inequality in expectation for controlled Markov diffusion processes is presented. Finally, in Chapter 8, we give conclusions and discuss potential future extensions of the developed research.

Chapter 2

Lyapunov and Converse Lyapunov Theorems for Stochastic Semistability

2.1. Introduction

Using a notion of stochastic semistability, almost sure consensus of multiagent systems under distributed nonlinear protocols over random networks, wherein the evolution of each link of the random network follows a Markov process, is addressed in [119]. In this chapter, we extend the notion of semistability to nonlinear stochastic systems that have a continuum of equilibrium solutions. In particular, we develop almost sure convergence and stochastic Lyapunov stability properties to address almost sure semistability requiring the trajectories of a nonlinear stochastic dynamical system to converge almost surely to a set of equilibrium solutions, wherein every equilibrium solution in the set is almost surely Lyapunov stable. Furthermore, we provide necessary and sufficient Lyapunov conditions for semistability and show that semistability implies the existence of a continuous Lyapunov function whose infinitesimal generator decreases along the dynamical system trajectories and is such that the Lyapunov function satisfies inequalities involving the average distance to the set of equilibria.

2.2. Notation, Definitions, and Mathematical Preliminaries

In this section, we establish notation, definitions, and develop mathematical preliminaries necessary for developing the results in this dissertation. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{Z}_+$ denotes the set of positive integers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, $\mathbb{N}^n$ denotes the set of $n\times n$ nonnegative-definite matrices, and $\mathbb{P}^n$ denotes the set of $n \times n$ positive-definite matrices. We write $\mathcal{B}_\varepsilon(x)$ for the *open ball centered* at x with *radius* ε, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenious matrix norm, A^{T} for the transpose of the matrix A, $\otimes$ for the Kronecker product, $\oplus$ for the Kronecker sum, and I_n or I for the $n \times n$ identity matrix. Furthermore, $\mathfrak{B}^n$ denotes the σ-algebra of Borel sets in $\mathcal{D} \subseteq \mathbb{R}^n$ and $\mathfrak{S}$ denotes a σ-algebra generated on a set $\mathcal{S} \subseteq \mathbb{R}^n$.

We define a complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω denotes the sample space, $\mathcal{F}$ denotes a σ-algebra, and $\mathbb{P}$ defines a probability measure on the σ-algebra $\mathcal{F}$; that is, $\mathbb{P}$ is a nonnegative countably additive set function on $\mathcal{F}$ such that $\mathbb{P}(\Omega) = 1$ [6]. Furthermore, we assume that $w(\cdot)$ is a standard d-dimensional Wiener process defined by $(w(\cdot), \Omega, \mathcal{F}, \mathbb{P}^{w_0})$, where $\mathbb{P}^{w_0}$ is the classical Wiener measure [83, p. 10], with a continuous-time filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by the Wiener process $w(t)$ up to time t. We denote a stochastic dynamical system by $\mathcal{G}$ generating a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ adapted to the stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ satisfying $\mathcal{F}_\tau \subset \mathcal{F}_t$, $0 \leq \tau < t$, such that $\{\omega \in \Omega : x(t, \omega) \in \mathcal{B}\} \in \mathcal{F}_t$, $t \geq 0$, for all Borel sets $\mathcal{B} \subset \mathbb{R}^n$ contained in the Borel σ-algebra $\mathfrak{B}^n$. Here we use the notation $x(t)$ to represent the stochastic process $x(t, \omega)$ omitting its dependence on ω.

We denote the set of equivalence classes of measurable, integrable, and square-integrable $\mathbb{R}^n$ or $\mathbb{R}^{n\times m}$ (depending on context) valued random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ over the semi-infinite parameter space $[0, \infty)$ by $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, respectively, where the equivalence relation is the one induced by $\mathbb{P}$-almost-sure equality. In particular, elements of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ take finite values $\mathbb{P}$-almost surely (a.s.). Hence, depending on the context, $\mathbb{R}^n$ will denote either the set of $n \times 1$ real variables or the subspace of $\mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ comprising of $\mathbb{R}^n$ random processes that are constant almost surely. All inequalities and equalities involving random processes on $(\Omega, \mathcal{F}, \mathbb{P})$ are to be understood to hold $\mathbb{P}$-almost surely. Furthermore, $\mathbb{E}[\,\cdot\,]$ and $\mathbb{E}^{x_0}[\,\cdot\,]$ denote, respectively, the expectation with respect to the probability measure $\mathbb{P}$ and with respect to the classical Wiener measure $\mathbb{P}^{x_0}$.

Given $x \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$, $\{x = 0\}$ denotes the set $\{\omega \in \Omega : x(t, \omega) = 0\}$, and so on. Given $x \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{E} \in \mathcal{F}$, we say x is nonzero on $\mathcal{E}$ if $\mathbb{P}(\{x = 0\} \cap \mathcal{E}) = 0$. Furthermore, given $x \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and a σ-algebra $\mathcal{E} \subseteq \mathcal{F}$, $\mathbb{E}^{\mathbb{P}}[x]$ and $\mathbb{E}^{\mathbb{P}}[x|\mathcal{E}]$ denote, respectively, the expectation of the random variable x and the conditional expectation of x given $\mathcal{E}$, with all moments taken under the measure $\mathbb{P}$. Here, for simplicity of exposition, we omit the symbol $\mathbb{P}$ in denoting expectation, and similarly for conditional expectation. Specifically, we denote the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by $\mathbb{E}[\,\cdot\,]$, and similarly for conditional expectation.

A stochastic process $x : \overline{\mathbb{R}}_+ \times \Omega \to \mathcal{D}$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ is called a *martingale* with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if and only if $x(t)$ is a $\mathcal{F}_t$-measurable random vector for all $t \geq 0$, $\mathbb{E}[x(t)] < \infty$, and $x(\tau) = \mathbb{E}[x(t)|\mathcal{F}_\tau]$ for all $t \geq \tau \geq 0$. If we replace the equality in the above equation with "$\leq$" (resp., "$\geq$"), then $x(\cdot)$ is a *supermartingale* (resp., *submartingale*). A random variable $\tau : \Omega \to [0, \infty]$ is called a *stopping time* with respect to $\mathcal{F}_t$ if and only if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$, $t \geq 0$.

Finally, we write $\operatorname{tr}(\cdot)$ for the trace operator, $(\cdot)^{-1}$ for the inverse operator, $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $V''(x) \triangleq \frac{\partial^2 V(x)}{\partial x^2}$ for the Hessian of V at x, and $\mathcal{H}_n$ for the Hilbert space of random vectors $x \in \mathbb{R}^n$ with finite average power, that is, $\mathcal{H}_n \triangleq \{x : \Omega \to \mathbb{R}^n : \mathbb{E}[x^{\mathrm{T}} x] < \infty\}$. For an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{H}_n^{\mathcal{D}} \triangleq \{x \in \mathcal{H}_n : x : \Omega \to \mathcal{D}\}$ denotes the set of all the random vectors in $\mathcal{H}_n$ induced by $\mathcal{D}$. Similarly, for every $x_0 \in \mathbb{R}^n$, $\mathcal{H}_n^{x_0} \triangleq \{x \in \mathcal{H}_n : x \overset{\text{a.s.}}{=} x_0\}$. Furthermore, C^2 denotes the space of real-valued functions $V : \mathcal{D} \to \mathbb{R}$ that are two-times continuously differentiable with respect to $x \in \mathcal{D} \subseteq \mathbb{R}^n$.

Definition 2.1 [63]. Let $(S, \mathfrak{S})$ and $(T, \mathfrak{T})$ be measurable spaces, and let $\mu : S \times \mathfrak{T} \to \overline{\mathbb{R}}_+$. If the function $\mu(s, B)$ is $\mathfrak{S}$-measurable in $s \in S$ for a fixed $B \in \mathfrak{T}$ and $\mu(s, B)$ is a probability measure in $B \in \mathfrak{T}$ for a fixed $s \in S$, then μ is called a (*probability*) *kernel* from S to T.

Furthermore, for $s \leq t$, the function $\mu_{s,t} : S \times \mathfrak{S} \to \mathbb{R}$ is called a *regular conditional probability measure* if $\mu_{s,t}(\cdot, \mathfrak{S})$ is measurable, $\mu_{s,t}(S, \cdot)$ is a probability measure, and $\mu_{s,t}(\cdot, \cdot)$ satisfies

$$\mu_{s,t}(x(s), B) = \mathbb{P}(x(t) \in B | x(s)) = \mathbb{P}(x(t) \in B | \mathcal{F}_s) \quad \text{a.s.}, \quad x(\cdot) \in \mathcal{H}_n. \tag{2.1}$$

Any family of regular conditional probability measures $\{\mu_{s,t}\}_{s \leq t}$ satisfying the Chapman-Kolmogorov equation [6] is called a *semigroup of Markov kernels*. The Markov kernels are called *time homogeneous* if and only if $\mu_{s,t} = \mu_{0,t-s}$ holds for all $s \leq t$.

Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = f(x(t))\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \qquad t \in \mathcal{I}_{x(0)}, \tag{2.2}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \mathcal{D}$, $w(\cdot)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, $x(0)$ is independent of $(w(t) - w(0)), t \geq 0$, $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n \times d}$ are continuous, $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0 \text{ and } D(x) = 0\}$ is nonempty, and $\mathcal{I}_{x(0)} = [0, \tau_{x(0)})$, $0 \leq \tau_{x(0)} \leq \infty$, is the maximal interval of existence for the solution $x(\cdot)$ of (2.2). An *equilibrium point* of (2.2) is a point $x_{\mathrm{e}} \in \mathbb{R}^n$ such that $f(x_{\mathrm{e}}) = 0$ and $D(x_{\mathrm{e}}) = 0$. It is easy to see that x_{e} is an equilibrium point of (2.2) if and only if the constant stochastic process $x(\cdot) \overset{\text{a.s.}}{=} x_{\mathrm{e}}$ is a solution of (2.2). We denote the set of equilibrium points of (2.2) by $\mathcal{E} \triangleq \{\omega \in \Omega : x(t, \omega) = x_{\mathrm{e}}\} = \{x_{\mathrm{e}} \in \mathcal{D} : f(x_{\mathrm{e}}) = 0 \text{ and } D(x_{\mathrm{e}}) = 0\}$.

The filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is clearly a real vector space with addition and scalar multiplication defined componentwise and pointwise. A $\mathbb{R}^n$-valued stochastic process $x : [0, \tau] \times \Omega \to \mathcal{D}$ is said to be a *solution* of (2.2) on the time interval $[0, \tau]$ with initial condition $x(0) \overset{\text{a.s.}}{=} x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, $f \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$x(t) = x_0 + \int_0^t f(x(\sigma))\mathrm{d}\sigma + \int_0^t D(x(\sigma))\mathrm{d}w(\sigma) \ \text{ a.s.}, \quad t \in [0, \tau], \tag{2.3}$$

where the integrals in (2.3) are Itô integrals.

Note that for each fixed $t \geq 0$, the random variable $\omega \mapsto x(t, \omega)$ assigns a vector $x(\omega)$ to every outcome $\omega \in \Omega$ of an experiment, and for each fixed $\omega \in \Omega$, the mapping $t \mapsto x(t, \omega)$ is the *sample path* of the stochastic process $x(t)$, $t \geq 0$. A pathwise solution $t \mapsto x(t)$ of (2.2) in $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{x_0})$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal pathwise solutions to (2.2) in $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}^{x_0})$ exist on $[0, \infty)$, and hence, we assume (2.2) is *forward complete*. Sufficient conditions for forward completeness or *global solutions* of (2.2) are given in [76].

Furthermore, we assume that $f : \mathcal{D} \to \mathbb{R}^n$ and $D : \mathcal{D} \to \mathbb{R}^{n\times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\mathrm{F}} \leq L\|x - y\|, \quad x, y \in \mathcal{D}\backslash\{0\}, \tag{2.4}$$

and the growth restriction condition

$$\|f(x)\|^2 + \|D(x)\|_{\mathrm{F}}^2 \leq L^2(1 + \|x\|^2), \quad x \in \mathcal{D}\backslash\{0\}, \tag{2.5}$$

for some Lipschitz constant $L > 0$, and hence, since $x(0) \in \mathcal{H}_n^{\mathcal{D}}$ and $x(0)$ is independent of $(w(t) - w(0)), t \geq 0$, it follows that there exists a unique solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ of (2.2) forward in time for all initial conditions in the following sense. For every $x \in \mathcal{H}_n^{\mathcal{D}}\backslash\{0\}$ there exists $\tau_x > 0$ such that, if $x_1 : [0, \tau_1] \times \Omega \to \mathcal{D}$ and $x_2 : [0, \tau_2] \times \Omega \to \mathcal{D}$ are two solutions of (2.2); that is, if $x_1, x_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, with continuous sample paths almost surely, solve (2.2), then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $\mathbb{P}\big(x_1(t) = x_2(t),\ 0 \leq t \leq \tau_x\big) = 1$.

A weaker sufficient condition for the existence of a unique solution to (2.2) using a notion of (finite or infinite) escape time under the local Lipschitz continuity condition (2.4) without the growth condition (2.5) is given in [113]. Moreover, the unique solution determines a $\mathbb{R}^n$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary

Feller transition probability function is given by $\big($[67, Thm. 3.4], [6, Thm. 9.2.8]$\big)$

$$\mathbb{P}(x(t) \in B | x(t_0) \overset{\text{a.s.}}{=} x_0) = \mathbb{P}(0, x_0, t - t_0, B), \quad x_0 \in \mathbb{R}^n, \tag{2.6}$$

for all $t \geq t_0$ and all Borel subsets $\mathcal{B}$ of $\mathbb{R}^n$, where $\mathbb{P}(\sigma, x, t, \mathcal{B}), t \geq \sigma$, denotes the probability of transition of the point $x \in \mathbb{R}^n$ at time instant s into the set $\mathcal{B} \subset \mathbb{R}^n$ at time instant t. Recall that every continuous process with Feller transition probability function is also a strong Markov process [67, p. 101]. Finally, we say that the dynamical system (2.2) is *convergent in probability* with respect to the closed set $\mathcal{H}_n^{\mathcal{D}_c} \subseteq \mathcal{H}_n^{\mathcal{D}}$ if and only if the pointwise $\lim_{t \to \infty} s(t, x)$ exists for every $x \in \mathcal{H}_n^{\mathcal{D}_c}$.

Definition 2.2. A point $p \in \mathcal{D}$ is a *limit point* of the trajectory $s(\cdot, x)$ of (2.2) if there exists a monotonic sequence $\{t_n\}_{n=0}^{\infty}$ of positive numbers, with $t_n \to \infty$ as $n \to \infty$, such that $s(t_n, x) \overset{\text{a.s.}}{\to} p$ as $n \to \infty$. The set of all limit points of $s(t, x), t \geq 0$, is the *limit set* $\omega(x)$ of $s(\cdot, x)$ of (2.2).

Definition 2.3 [83, Def. 7.7]. Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_n^{\mathcal{D}}$ and let $V : \mathcal{D} \to \mathbb{R}$. Then the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq 0$, with $x(0) \overset{\text{a.s.}}{=} x_0$, is defined by

$$\mathcal{L}V(x_0) \triangleq \lim_{t \to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{D}, \tag{2.7}$$

where $\mathbb{E}^{x_0}$ denotes the expectation with respect to the probability measure $\mathbb{P}^{x_0}(x(t) \in \mathcal{B}) \triangleq \mathbb{P}(0, x_0, t, \mathcal{B})$.

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geq 0$, satisfies (2.2), then the limit in (2.7) exists for all $x \in \mathcal{D}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geq 0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (2.2) and is given by ([83, Thm. 7.9])

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\; D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x), \quad x \in \mathcal{D}. \tag{2.8}$$

Definition 2.4 [75]. An open set $\mathcal{D} \subset \mathbb{R}^n$ is said to be *invariant with respect to* (2.2) if, for all $x_0 \in \mathcal{D}$, $\mathbb{P}\left(x(t) \in \mathcal{D}\right) = 1$, $t \geq 0$.

The following definition introduces several notions of stability in probability.

Definition 2.5 [67, 75]. *i*) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is *Lyapunov stable in probability* if, for every $\varepsilon > 0$,

$$\lim_{x_0 \to x_{\text{e}}} \mathbb{P}^{x_0}\left(\sup_{t \geq 0} \|x(t) - x_{\text{e}}\| > \varepsilon\right) = 0. \tag{2.9}$$

Equivalently, the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is Lyapunov stable in probability if, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\rho, \varepsilon) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_{\text{e}})$,

$$\mathbb{P}^{x_0}\left(\sup_{t \geq 0} \|x(t) - x_{\text{e}}\| > \varepsilon\right) \leq \rho. \tag{2.10}$$

ii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is *asymptotically stable in probability* if it is Lyapunov stable in probability and

$$\lim_{x_0 \to x_{\text{e}}} \mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t) - x_{\text{e}}\| = 0\right) = 1. \tag{2.11}$$

Equivalently, the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is asymptotically stable in probability if it is Lyapunov stable in probability and, for every $\rho \in (0, 1)$, there exist $\delta = \delta(\rho) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_{\text{e}})$, then

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t) - x_{\text{e}}\| = 0\right) \geq 1 - \rho. \tag{2.12}$$

iii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is *globally asymptotically stable in probability* if it is Lyapunov stable in probability and, for all $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x(t)\| = 0\right) = 1. \tag{2.13}$$

iv) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is *exponentially p-stable in probability* if there exist scalars α, β, and $\delta > 0$, and $p \geq 1$ such that if $x_0 \in \mathcal{B}_\delta(x_{\text{e}})$, then

$$\mathbb{E}^{x_0}\left[\|x(t)\|^p\right] \leq \alpha \|x_0\|^p e^{-\beta t}. \tag{2.14}$$

If, in addition, (2.14) holds for all $x_0 \in \mathbb{R}^n$, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is *globally exponentially p-stable in probability*. Finally, if $p = 2$, then we say that the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is *(globally) exponentially mean square stable in probability*.

The following lemma gives an equivalent characterization of Lyapunov and asymptotic stability in probability in terms of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions. For the definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$, and $\mathcal{KL}$ functions see [45, p.162].

Lemma 2.1. *i*) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is Lyapunov stable in probability if and only if for every $\rho > 0$ there exist a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ and a constant $c = c(\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_c(x_{\text{e}})$,

$$\mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \alpha_\rho(\|x_0 - x_{\text{e}}\|)\right) \le \rho, \quad t \ge 0. \tag{2.15}$$

ii) The equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_{\text{e}}$ to (2.2) is asymptotically stable in probability if and only if for every $\rho > 0$ there exist a class $\mathcal{KL}$ function $\beta_\rho(\cdot,\cdot)$ and a constant $c = c(\rho) > 0$ such that, for all $x_0 \in \mathcal{B}_c(x_{\text{e}})$,

$$\mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \beta_\rho(\|x_0 - x_{\text{e}}\|, t)\right) \le \rho, \quad t \ge 0. \tag{2.16}$$

Proof. *i*) Suppose there exist a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ and a constant $c = c(\rho) > 0$ such that, for every $\rho > 0$ and $x_0 \in \mathcal{B}_c(x_{\text{e}})$,

$$\mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \alpha_\rho(\|x_0 - x_{\text{e}}\|)\right) \le \rho, \quad t \ge 0. \tag{2.17}$$

Now, given $\varepsilon > 0$, let $\delta(\rho, \varepsilon) = \min\{c(\rho), \alpha_\rho^{-1}(\varepsilon)\}$. Then, for $x_0 \in \mathcal{B}_\delta(x_{\text{e}})$ and $t \ge 0$,

$$\begin{aligned}
\mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \alpha_\rho(\|x_0 - x_{\text{e}}\|)\right) &\ge \mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \alpha_\rho(\delta)\right) \\
&\ge \mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \alpha_\rho(\alpha_\rho^{-1}(\varepsilon))\right) \\
&\ge \mathbb{P}^{x_0}\left(\|x(t) - x_{\text{e}}\| > \varepsilon\right).
\end{aligned}$$

Therefore, for every given $\varepsilon > 0$ and $\rho > 0$, there exists $\delta > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$,

$$\mathbb{P}^{x_0}\left(\sup_{t\geq 0}\|x(t) - x_e\| > \varepsilon\right) \leq \rho,$$

which proves that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is Lyapunov stable in probability.

Conversely, for every given ε and ρ, let $\bar{\delta}(\varepsilon,\rho)$ be the supremum of all admissible $\delta(\varepsilon,\rho)$. Note that the function $\delta(\cdot,\cdot)$ is positive and nondecreasing in its first argument, but not necessarily continuous. For every $\rho > 0$ chose a class $\mathcal{K}$ function $\gamma_\rho(r)$ such that $\gamma_\rho(r) \leq k\bar{\delta}(r,\rho)$, $0 < k < 1$. Let $c(\rho) = \lim_{r\to\infty}\gamma_\rho(r)$ and $\alpha_\rho(r) = \gamma_\rho^{-1}(r)$, and note that $\alpha_\rho(\cdot)$ is class $\mathcal{K}$ [66, Lemma 4.2]. Next, for every $\rho > 0$ and $x_0 \in \mathcal{B}_{c(\rho)}(x_e)$, let $\varepsilon = \alpha_\rho(\|x_0 - x_e\|)$. Then, $\|x_0 - x_e\| < \bar{\delta}(\varepsilon,\rho)$ and

$$\mathbb{P}^{x_0}\left(\sup_{t\geq 0}\|x(t) - x_e\| > \varepsilon\right) \leq \rho \tag{2.18}$$

implies

$$\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right) \leq \rho, \quad t \geq 0. \tag{2.19}$$

ii) Suppose there exists a class $\mathcal{KL}$ function $\beta(r,s)$ such that (2.16) is satisfied. Then,

$$\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \beta_\rho(\|x_0 - x_e\|, 0)\right) \leq \rho, \quad t \geq 0,$$

which implies that equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is Lyapunov stable in probability. Moreover, for $x_0 \in \mathcal{B}_{c(\rho)}(x_e)$, the solution to (2.2) satisfies

$$\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \beta_\rho(\|c(\rho)\|, t)\right) \leq \rho, \quad t \geq 0.$$

Now, letting $t \to \infty$ yields $\mathbb{P}^{x_0}\left(\lim_{t\to\infty}\|x(t) - x_e\| > 0\right) \leq \rho$ for every $\rho > 0$, and hence, $\mathbb{P}^{x_0}\left(\lim_{t\to\infty}\|x(t) - x_e\| = 0\right) \geq 1 - \rho$, which implies that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is asymptotically stable in probability.

Conversely, suppose that the equilibrium solution $x(t) \stackrel{\text{a.s.}}{\equiv} x_e$ is asymptotically stable in probability. In this case, for every $\rho > 0$ there exist a constant $c(\rho) > 0$ and a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ such that, for every $r \in (0, c(\rho)]$, the solution $x(t)$, $t \geq 0$, to (2.2) satisfies

$$\mathbb{P}^{x_0}\left(\sup_{t\geq 0}\|x(t) - x_e\| > \alpha_\rho(r)\right) \leq \mathbb{P}^{x_0}\left(\sup_{t\geq 0}\|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right) \leq \rho \tag{2.20}$$

for all $\|x_0 - x_e\| < r$. Moreover, given $\eta > 0$ there exists $T = T_\rho(\eta, r) \geq 0$ such that

$$\mathbb{P}^{x_0}\left(\sup_{t \geq T_\rho(\eta,r)} \|x(t) - x_e\| > \eta\right) \leq \rho.$$

Let $\overline{T}_\rho(\eta, r)$ be the infimum of all admissible $T_\rho(\eta, r)$ and note that $\overline{T}_\rho(\eta, r)$ is nonnegative and nonincreasing in η, nondecreasing in r, and $\overline{T}_\rho(\eta, r) = 0$ for all $\eta \geq \alpha(r)$. Now, let

$$W_{r,\rho}(\eta) = \frac{2}{\eta}\int_{\frac{\eta}{2}}^{\eta} \overline{T}_\rho(s, r)\mathrm{d}s + \frac{r}{\eta} \geq \overline{T}_\rho(\eta, r) + \frac{r}{\eta}$$

and note that $W_{r,\rho}(\eta)$ is positive and has the following properties: i) For every fixed r and ρ, $W_{r,\rho}(\eta)$ is continuous, strictly decreasing, and $\lim_{\eta\to\infty} W_{r,\rho}(\eta) = 0$; and ii) for every fixed η and ρ, $W_{r,\rho}(\eta)$ is strictly increasing in r.

Next, let $U_{r,\rho} = W_{r,\rho}^{-1}$ and note that $U_{r,\rho}$ satisfies properties i) and ii) of $W_{r,\rho}$, and $\overline{T}_\rho(U_{r,\rho}(\sigma), r) < W_{r,\rho}(U_{r,\rho}(\sigma)) = \sigma$. Therefore,

$$\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > U_{r,\rho}(t)\right) \leq \rho, \quad t \geq 0, \tag{2.21}$$

for all $\|x_0 - x_e\| < r$. Now, using (2.20) and (2.21) it follows that

$$\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \sqrt{\alpha_\rho(\|x_0 - x_e\|)U_{c(\rho),\rho}(t)}\right) \leq \rho, \quad \|x_0 - x_e\| < c(\rho), \quad t \geq 0.$$

Thus, inequality (2.16) is satisfied with $\beta_\rho(\|x_0 - x_e\|, t) = \sqrt{\alpha_\rho(\|x_0 - x_e\|)U_{c(\rho),\rho}(t)}$. □

The following proposition gives a sufficient condition for a trajectory of (2.2) to converge to a limit point. For this result, $\mathcal{D}_c \subseteq \mathcal{D}$ denotes a positively invariant set with respect to (2.2) and $s_t(\mathcal{H}_n^{\mathcal{D}_c})$ denotes the image of $\mathcal{H}_n^{\mathcal{D}_c} \subset \mathcal{H}_n^{\mathcal{D}}$ under the flow $s_t : \mathcal{H}_n^{\mathcal{D}_c} \to \mathcal{H}_n^{\mathcal{D}}$; that is, $s_t(\mathcal{H}_n^{\mathcal{D}_c}) \triangleq \{y : y = s_t(x_0) \text{ for some } x(0) \overset{\text{a.s.}}{=} x_0 \in \mathcal{H}_n^{\mathcal{D}_c}\}$.

Proposition 2.1. Consider the nonlinear stochastic dynamical system (2.2) and let $x \in \mathcal{H}_n^{\mathcal{D}_c}$. If the limit set $\omega(x)$ of (2.2) contains a Lyapunov stable in probability equilibrium point y, then $y \overset{\text{a.s.}}{=} \lim_{t\to\infty} s(t, x)$ as $x \to y$, that is, $\omega(x) \overset{\text{a.s.}}{=} \{y\}$ as $x \to y$.

Proof. Suppose $y \in \omega(x)$ is Lyapunov stable in probability and let $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_c$ be an open neighborhood of y. Since y is Lyapunov stable in probability, there exists an open neighborhood $\mathcal{N}_\delta \subset \mathcal{D}_c$ of y such that $s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ as $x \to y$ for every $t \geq 0$. Now, since $y \in \omega(x)$, it follows that there exists $\tau \geq 0$ such that $s(\tau, x) \in \mathcal{H}_n^{\mathcal{N}_\delta}$. Hence, $s(t+\tau, x) = s_t(s(\tau, x)) \in s_t(\mathcal{H}_n^{\mathcal{N}_\delta}) \subseteq \mathcal{H}_n^{\mathcal{N}_\varepsilon}$ for every $t > 0$. Since $\mathcal{N}_\varepsilon \subseteq \mathcal{D}_c$ is arbitrary, it follows that $y \overset{\text{a.s.}}{=} \lim_{t\to\infty} s(t, x)$. Thus, $\lim_{n\to\infty} s(t_n, x) \overset{\text{a.s.}}{=} y$ as $x \to y$ for every sequence $\{t_n\}_{n=1}^\infty$, and hence, $\omega(x) \overset{\text{a.s.}}{=} \{y\}$ as $x \to y$. □

The following definition introduces the notion of stochastic semistability. For the statement of this definition define $\operatorname{dist}(x, \mathcal{E}) \triangleq \inf_{y\in\mathcal{E}} \|x - y\|$.

Definition 2.6. An equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ of (2.2) is *stochastically semistable* if the following statements hold.

i) For every $\varepsilon > 0$, $\lim_{x_0\to x_e} \mathbb{P}^{x_0}\left(\sup_{0\leq t<\infty} \|x(t) - x_e\| > \varepsilon\right) = 0$. Equivalently, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $x_0 \in \mathcal{B}_\delta(x_e)$, $\mathbb{P}^{x_0}\left(\sup_{0\leq t<\infty} \|x(t) - x_e\| > \varepsilon\right) \leq \rho$.

ii) $\lim_{\operatorname{dist}(x_0,\mathcal{E})\to 0} \mathbb{P}^{x_0}(\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0) = 1$. Equivalently, for every $\rho > 0$, there exist $\delta = \delta(\rho) > 0$ such that if $\operatorname{dist}(x_0, \mathcal{E}) \leq \delta$, then $\mathbb{P}^{x_0}(\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0) \geq 1 - \rho$.

The dynamical system (2.2) is *stochastically semistable* if every equilibrium solution of (2.2) is stochastically semistable. Finally, the dynamical system (2.2) is *globally stochastically semistable* if *i)* holds and $\mathbb{P}^{x_0}(\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0) = 1$ for all $x_0 \in \mathbb{R}^n$.

Remark 2.1. If $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ only satisfies *i)* in Definition 2.6, then the equilibrium solution $x(t) \overset{\text{a.s.}}{\equiv} x_e \in \mathcal{E}$ of (2.2) is Lyapunov stable in probability.

Definition 2.7. For $\rho \in (0, 1)$, the *ρ-domain of semistability* is the set of points $x_0 \in \mathcal{D}$ such that if $x(t)$, $t \geq 0$, is a solution to (2.2) with $x(0) \overset{\text{a.s.}}{=} x_0$, then $x(t)$ converges to a Lyapunov stable in probability equilibrium point in $\mathcal{D}$ with probability greater than or equal to $1 - \rho$.

Note that if (2.2) is stochastically semistable, then its ρ-domain of semistability contains the set of equilibria in its interior. Next, we present alternative equivalent characterizations for stochastic semistability of (2.2).

Proposition 2.2. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (2.2). Then the following statements are equivalent:

i) $\mathcal{G}$ is stochastically semistable.

ii) For every $x_{\rm e} \in \mathcal{E}$ and $\rho > 0$, there exist class $\mathcal{K}$ and $\mathcal{L}$ functions $\alpha_\rho(\cdot)$ and $\beta_\rho(\cdot)$, respectively, and $\delta = \delta(x_{\rm e}, \rho) > 0$ such that, if $x_0 \in \mathcal{B}_\delta(x_{\rm e})$, then

$$\mathbb{P}^{x_0}\left(\|x(t) - x_{\rm e}\| > \alpha_\rho(\|x_0 - x_{\rm e}\|)\right) \le \rho, \quad t \ge 0$$

and $\mathbb{P}^{x_0}\left(\operatorname{dist}(x(t), \mathcal{E}) > \beta_\rho(t)\right) \le \rho$, $t \ge 0$.

iii) For every $x_{\rm e} \in \mathcal{E}$ and $\rho > 0$, there exist class $\mathcal{K}$ functions $\alpha_{1\rho}(\cdot)$ and $\alpha_{2\rho}(\cdot)$, a class $\mathcal{L}$ function $\beta_\rho(\cdot)$, and $\delta = \delta(x_{\rm e}, \rho) > 0$ such that, if $x_0 \in \mathcal{B}_\delta(x_{\rm e})$, then

$$\begin{aligned}\mathbb{P}^{x_0}&\left(\operatorname{dist}(x(t), \mathcal{E}) > \alpha_{2\rho}(\|x_0 - x_{\rm e}\|)\beta_\rho(t)\right)\\ &\le \mathbb{P}^{x_0}\left(\alpha_{1\rho}(\|x(t) - x_{\rm e}\|) > \alpha_{2\rho}(\|x_0 - x_{\rm e}\|)\right) \le \rho, \quad t \ge 0.\end{aligned}$$

Proof. To show that *i*) implies *ii*), suppose (2.2) is stochastically semistable and let $x_{\rm e} \in \mathcal{E}$. It follows from Lemma 2.1 that for every $\rho > 0$ there exists $\delta = \delta(x_{\rm e}, \rho) > 0$ and a class $\mathcal{K}$ function $\alpha_\rho(\cdot)$ such that if $\|x_0 - x_{\rm e}\| \le \delta$, then $\mathbb{P}^{x_0}\left(\|x(t) - x_{\rm e}\| > \alpha_\rho(\|x_0 - x_{\rm e}\|)\right) \le \rho$, $t \ge 0$. Without loss of generality, we may assume that δ is such that $\overline{\mathcal{B}_\delta(x_{\rm e})}$ is contained in the ρ-domain of semistability of (2.2). Hence, for every $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, $\lim_{t\to\infty} x(t) \overset{\rm a.s.}{=} x^* \in \mathcal{E}$ and, consequently, $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \operatorname{dist}(x(t), \mathcal{E}) = 0\right) = 1$.

For every $\varepsilon > 0$, $\rho > 0$, and $x_0 \in \overline{\mathcal{B}_\delta(x_{\rm e})}$, define $T_{x_0}(\varepsilon, \rho)$ to be the infimum of T with the property that $\mathbb{P}^{x_0}\left(\sup_{t\ge T} \operatorname{dist}(x(t), \mathcal{E}) > \varepsilon\right) \le \rho$, that is,

$$T_{x_0}(\varepsilon, \rho) \triangleq \inf\left\{T : \mathbb{P}^{x_0}\left(\sup_{t\ge T} \operatorname{dist}(x(t), \mathcal{E}) > \varepsilon\right) \le \rho\right\}.$$

For each $x_0 \in \overline{\mathcal{B}_\delta(x_e)}$ and ρ, the function $T_{x_0}(\varepsilon,\rho)$ is nonnegative and nonincreasing in ε, and $T_{x_0}(\varepsilon,\rho) = 0$ for sufficiently large ε.

Next, let $T(\varepsilon,\rho) \triangleq \sup\{T_{x_0}(\varepsilon,\rho) : x_0 \in \overline{\mathcal{B}_\delta(x_e)}\}$. We claim that T is well defined. To show this, consider $\varepsilon > 0$, $\rho > 0$, and $x_0 \in \overline{\mathcal{B}_\delta(x_e)}$. Since $\mathbb{P}^{x_0}\left(\sup_{t \geq T_{x_0}(\varepsilon,\rho)} \mathrm{dist}(x(t),\mathcal{E}) > \varepsilon\right) \leq \rho$, it follows from the sample continuity of s that, for every $\varepsilon > 0$ and $\rho > 0$, there exists an open neighborhood $\mathcal{U}$ of x_0 such that $\mathbb{P}^{x_0}\left(\sup_{t \geq T_z(\varepsilon,\rho)} \mathrm{dist}(s(t,z),\mathcal{E}) > \varepsilon\right) \leq \rho$ for every $z \in \mathcal{U}$. Hence, $\limsup_{z \to x_0} T_z(\varepsilon,\rho) \leq T_{x_0}(\varepsilon,\rho)$ implying that the function $x_0 \mapsto T_{x_0}(\varepsilon,\rho)$ is upper semicontinuous at the arbitrarily chosen point x_0, and hence on $\overline{\mathcal{B}_\delta(x_e)}$. Since an upper semicontinuous function defined on a compact set achieves its supremum, it follows that $T(\varepsilon,\rho)$ is well defined. The function $T(\cdot)$ is the pointwise supremum of a collection of nonnegative and nonincreasing functions, and hence is nonnegative and nonincreasing. Moreover, $T(\varepsilon,\rho) = 0$ for every $\varepsilon > \max\{\alpha_\rho(\|x_0 - x_e\|) : x_0 \in \overline{\mathcal{B}_\delta(x_e)}\}$.

Let $\psi_\rho(\varepsilon) \triangleq \frac{2}{\varepsilon}\int_{\varepsilon/2}^{\varepsilon} T(\sigma,\rho)\mathrm{d}\sigma + \frac{1}{\varepsilon} \geq T(\varepsilon,\rho) + \frac{1}{\varepsilon}$. The function $\psi_\rho(\varepsilon)$ is positive, continuous, strictly decreasing, and $\psi_\rho(\varepsilon) \to 0$ as $\varepsilon \to \infty$. Choose $\beta_\rho(\cdot) = \psi^{-1}(\cdot)$. Then $\beta_\rho(\cdot)$ is positive, continuous, strictly decreasing, and $\lim_{\sigma\to\infty} \beta_\rho(\sigma) = 0$. Furthermore, $T(\beta_\rho(\sigma),\rho) < \psi_\rho(\beta_\rho(\sigma)) = \sigma$. Hence, $\mathbb{P}^{x_0}\left(\mathrm{dist}(x(t),\mathcal{E}) > \beta_\rho(t)\right) \leq \rho$, $t \geq 0$.

Next, to show that *ii*) implies *iii*), suppose *ii*) holds and let $x_e \in \mathcal{E}$. Then it follows from *i*) of Lemma 2.1 that x_e is Lyapunov stable in probability. For every $\rho > 0$, choosing x_0 sufficiently close to x_e, it follows from the inequality $\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right) \leq \rho$, $t \geq 0$, that trajectories of (2.2) starting sufficiently close to x_e are bounded, and hence, the positive limit set of (2.2) is nonempty. Since $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \mathrm{dist}(x(t),\mathcal{E}) = 0\right) = 1$ as $\mathrm{dist}(x_0,\mathcal{E}) \to 0$, it follows that the positive limit set is contained in $\mathcal{E}$ as $\mathrm{dist}(x_0,\mathcal{E}) \to 0$. Now, since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} x(t) \overset{\text{a.s.}}{=} x^*$ as $x_0 \to x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. If $x^* = x_e$, then it follows using similar arguments as above that there exists a class $\mathcal{L}$ function $\hat{\beta}_\rho(\cdot)$ such that $\mathbb{P}^{x_0}\left(\mathrm{dist}(x(t),\mathcal{E}) > \hat{\beta}_\rho(t)\right) \leq \mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \hat{\beta}_\rho(t)\right) \leq \rho$ for every x_0

satisfying $\|x_0 - x_e\| < \delta$ and $t \geq 0$. Hence, $\mathbb{P}^{x_0}\left(\mathrm{dist}(x(t), \mathcal{E}) > \sqrt{\|x(t) - x_e\|}\sqrt{\hat{\beta}_\rho(t)}\right) \leq \rho$, $t \geq 0$. Next, consider the case where $x^* \neq x_e$ and let $\alpha_{1\rho}(\cdot)$ be a class $\mathcal{K}$ function. In this case, note that $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{E})/\alpha_{1\rho}(\|x(t) - x_e\|) = 0\right) \geq 1 - \rho$, and hence, it follows using similar arguments as above that there exists a class $\mathcal{L}$ function $\beta_\rho(\cdot)$ such that $\mathbb{P}^{x_0}\big(\mathrm{dist}(x(t), \mathcal{E}) > \alpha_{1\rho}(\|x(t) - x_e\|)\beta_\rho(t)\big) \leq \rho$, $t \geq 0$. Finally, note that $\alpha_{1\rho} \circ \alpha_\rho$ is of class $\mathcal{K}$ (by [66, Lemma 4.2]), and hence, *iii*) follows immediately.

Finally, to show that *iii*) implies *i*), suppose *iii*) holds and let $x_e \in \mathcal{E}$. Then it follows that for every $\rho > 0$, $\mathbb{P}^{x_0}\left(\alpha_{1\rho}(\|x(t) - x_e\|) > \alpha_{2\rho}(\|x(0) - x_e\|)\right) \leq \rho$, $t \geq 0$, that is, $\mathbb{P}^{x_0}[\|x(t) - x_e\| > \alpha_\rho(\|x(0) - x_e\|)] \leq \rho$, where $t \geq 0$ and $\alpha_\rho = {\alpha_{1\rho}}^{-1} \circ \alpha_{2\rho}$ is of class $\mathcal{K}$ (by [66, Lemma 4.2]). It now follows from *i*) of Lemma 2.1 that x_e is Lyapunov stable in probability. Since x_e was chosen arbitrarily, it follows that every equilibrium point is Lyapunov stable in probability. Furthermore, $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \mathrm{dist}(x(t), \mathcal{E}) = 0\right) \geq 1 - \rho$. Choosing x_0 sufficiently close to x_e, it follows from the inequality $\mathbb{P}^{x_0}\left(\|x(t) - x_e\| > \alpha_\rho(\|x_0 - x_e\|)\right) \leq \rho$, $t \geq 0$, that trajectories of (2.2) are almost sure bounded as $x_0 \to x_e$, and hence, the positive limit set of (2.2) is nonempty as $x_0 \to x_e$. Since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} x(t) \overset{\text{a.s.}}{=} x^*$ as $x_0 \to x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. Hence, by Definition 2.6, (2.2) is stochastically semistable. □

2.3. Stochastic Semistability of Nonlinear Dynamical Systems

In this section, we develop necessary and sufficient conditions for stochastic semistability. First, we present a sufficient condition for stochastic semistability.

Theorem 2.1. Consider the nonlinear stochastic dynamical system (2.2). Let $\mathcal{Q}$ be an open neighborhood of $\mathcal{E}$ and assume that there exists a two-times continuously differentiable function $V : \mathcal{Q} \to \overline{\mathbb{R}}_+$ such that

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathcal{Q}\backslash\mathcal{E}. \tag{2.22}$$

If every equilibrium point of (2.2) is Lyapunov stable in probability, then (2.2) is stochastically semistable. Moreover, if $\mathcal{Q} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then (2.2) is globally stochastically semistable.

Proof. Since every equilibrium point of (2.2) is Lyapunov stable in probability by assumption, for every $z \in \mathcal{E}$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z))$, $\varepsilon > 0$, is bounded and contained in $\mathcal{Q}$ as $\varepsilon \to 0$. The set $\mathcal{V}_\varepsilon \triangleq \bigcup_{z\in\mathcal{E}} \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$, $\varepsilon > 0$, is an open neighborhood of $\mathcal{E}$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}_\varepsilon$ so that there exists $z \in \mathcal{E}$ such that $x \in \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ and $s(t,x) \in \mathcal{H}_n^{\mathcal{V}_z\cap\mathcal{B}_\varepsilon(z)}$, $t \geq 0$, as $\varepsilon \to 0$. Since $\mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ is bounded and invariant with respect to the solution of (2.2) as $\varepsilon \to 0$, it follows that $\mathcal{V}_\varepsilon$ is invariant with respect to the solution of (2.2) as $\varepsilon \to 0$. Furthermore, it follows from (2.22) that $\mathcal{L}V(s(t,x)) < 0$, $t \geq 0$, and hence, since $\mathcal{V}_\varepsilon$ is bounded it follows from [75, Cor. 4.1] that $\lim_{t\to\infty} \mathcal{L}V(s(t,x)) \overset{\text{a.s.}}{=} 0$ as $\varepsilon \to 0$.

It is easy to see that $\mathcal{L}V(x) \neq 0$ by assumption and $\mathcal{L}V(x_{\rm e}) = 0$, $x_{\rm e} \in \mathcal{E}$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{E}$ as $t \to \infty$ and $\varepsilon \to 0$, which implies that $\lim_{\text{dist}(x,\mathcal{E})\to 0} \mathbb{P}^x(\lim_{t\to\infty} \text{dist}(s(t,x),\mathcal{E}) = 0) = 1$. Finally, since every point in $\mathcal{E}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$ as $x \to x^*$, where $x^* \in \mathcal{E}$ is Lyapunov stable in probability. Hence, by Definition 2.6, (2.2) is semistable. For $\mathcal{Q} = \mathbb{R}^n$ global stochastic semistability follows from identical arguments using the radially unbounded condition on $V(\cdot)$. □

Next, we present a slightly more general theorem for stochastic semistability wherein we do not assume that all points in $\mathcal{L}V^{-1}(0)$ are Lyapunov stable in probability but rather we assume that all points in $(\eta \circ V)^{-1}(0)$ are Lyapunov stable in probability for some continuous function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$.

Theorem 2.2. Consider the nonlinear stochastic dynamical system (2.2) and let $\mathcal{Q}$ be an open neighborhood of $\mathcal{E}$. Assume that there exists a two-times continuously differentiable function $V : \mathcal{Q} \to \overline{\mathbb{R}}_+$ and a continuous function $\eta : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ such that

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \leq -\eta(V(x)), \quad x \in \mathcal{Q}. \tag{2.23}$$

If every point in the set $\mathcal{M} \triangleq \{x \in \mathcal{Q} : \eta(V(x)) = 0\}$ is Lyapunov stable in probability, then (2.2) is stochastically semistable. Moreover, if $\mathcal{Q} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then (2.2) is globally stochastically semistable.

Proof. Since, by assumption, (2.2) is Lyapunov stable in probability for all $z \in \mathcal{M}$, there exists an open neighborhood $\mathcal{V}_z$ of z such that $s([0,\infty) \times \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z))$, $\varepsilon > 0$, is bounded and contained in $\mathcal{Q}$ as $\varepsilon \to 0$. The set $\mathcal{V}_\varepsilon \triangleq \bigcup_{z\in\mathcal{M}} \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ is an open neighborhood of $\mathcal{M}$ contained in $\mathcal{Q}$. Consider $x \in \mathcal{V}_\varepsilon$ so that there exists $z \in \mathcal{M}$ such that $x \in \mathcal{V}_z \cap \mathcal{B}_\varepsilon(z)$ and $s(t,x) \in \mathcal{H}_n^{\mathcal{V}_z\cap\mathcal{B}_\varepsilon(z)}$, $t \geq 0$, as $\varepsilon \to 0$. Since $\mathcal{V}_z$ is bounded it follows that $\mathcal{V}_\varepsilon$ is invariant with respect to the solution of (2.2) as $\varepsilon \to 0$. Furthermore, it follows from (2.23) that $\mathcal{L}V(s(t,x)) \leq -\eta(V(s(t,x)))$, $t \geq 0$, and hence, since $\mathcal{V}_\varepsilon$ is bounded and invariant with respect to the solution of (2.2) as $\varepsilon \to 0$, it follows from [75, Cor. 4.2] that $\lim_{t\to\infty} \eta(V(s(t,x))) \overset{\text{a.s.}}{=} 0$ as $\varepsilon \to 0$. Therefore, $s(t,x) \overset{\text{a.s.}}{\to} \mathcal{M}$ as $t \to \infty$ and $\varepsilon \to 0$, which implies that $\lim_{\mathrm{dist}(x,\mathcal{M})\to 0} \mathbb{P}^x\left(\lim_{t\to\infty} \mathrm{dist}(s(t,x),\mathcal{M}) = 0\right) = 1$. Finally, since every point in $\mathcal{M}$ is Lyapunov stable in probability, it follows from Proposition 2.1 that $\lim_{t\to\infty} s(t,x) \overset{\text{a.s.}}{=} x^*$ as $x \to x^*$, where $x^* \in \mathcal{M}$ is Lyapunov stable in probability. Hence, by definition, (2.2) is semistable. For $\mathcal{Q} = \mathbb{R}^n$ global stochastic semistability follows from identical arguments using the radially unbounded condition on $V(\cdot)$. □

Example 2.1. Consider the nonlinear stochastic dynamical system on $\mathcal{H}_2$ given by

$$\mathrm{d}x_1(t) = [\sigma_{12}(x_2(t)) - \sigma_{21}(x_1(t))]\mathrm{d}t + \gamma(x_2(t) - x_1(t))\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{2.24}$$

$$\mathrm{d}x_2(t) = [\sigma_{21}(x_1(t)) - \sigma_{12}(x_2(t))]\mathrm{d}t + \gamma(x_1(t) - x_2(t))\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{2.25}$$

where $\sigma_{ij}(\cdot)$, $i,j = 1,2$, $i \neq j$, ar continuous and $\gamma > 0$. Equations (2.24) and (2.25) represent the collective dynamics of two agents which interact by exchanging information.

The information states of the agents are described by the scalar random variables x_1 and x_2. The unity coefficients scaling $\sigma_{ij}(\cdot)$, $i, j \in \{1, 2\}$, $i \neq j$, appearing in (2.24) and (2.25) represent the topology of the information exchange between the agents. More specifically, given $i, j \in \{1, 2\}$, $i \neq j$, a coefficient of 1 denotes that agent j receives information from agent i, and a coefficient of zero denotes that agent i and j are disconnected, and hence, cannot share any information.

The communication topology between the agents can be represented by a graph $\mathfrak{G}$ having two nodes such that $\mathfrak{G}$ has a directed edge from node i to node j if and only if agent j can receive information from agent i. Since the coefficients scaling $\sigma_{ij}(\cdot)$, $i, j \in \{1, 2\}$, $i \neq j$, are constants, the communication topology is fixed. Furthermore, note that the directed communication graph $\mathfrak{G}$ is *weakly connected* since the underlying undirected graph is connected; that is, every agent receives information from, or delivers information to, at least one other agent.

Note that (2.24) and (2.25) can be cast in the form of (2.2) with

$$f(x) = \begin{bmatrix} \sigma_{12}(x_2) - \sigma_{21}(x_1) \\ \sigma_{21}(x_1) - \sigma_{12}(x_2) \end{bmatrix}, \quad D(x) = \begin{bmatrix} \gamma(x_2 - x_1) \\ \gamma(x_1 - x_2) \end{bmatrix},$$

where the stochastic term $D(x)\mathrm{d}w$ represents probabilistic variations in the information transfer between the agents. Furthermore, note that since

$$\mathbf{e}_2^{\mathrm{T}}\mathrm{d}x(t) = \mathbf{e}_2^{\mathrm{T}}f(x(t))\mathrm{d}t + \mathbf{e}_2^{\mathrm{T}}D(x(t))\mathrm{d}w(t) = 0, \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0,$$

where $\mathbf{e}_2 \triangleq [1\ 1]^{\mathrm{T}}$, it follows that $\mathrm{d}x_1(t) + \mathrm{d}x_2(t) = 0$, which implies that the total information is conserved.

In this example, we use Theorem 2.1 to analyze the collective behavior of (2.24) and (2.25). Specifically, we are interested in the consensus (i.e., state equipartitioning) behavior of the agents. For this purpose, we make the assumptions $\sigma_{ij}(x_j) - \sigma_{ji}(x_i) = 0$ if and only if $x_i = x_j$, $i \neq j$, and $(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)] \leq -\gamma^2(x_1 - x_2)$ for $i, j \in \{1, 2\}$. The first assumption implies that if the information (or energies) in the connected agents i and j

are equal, then information exchange between the agents is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. The second assumption implies that information flows from information rich agents to information poor agents and is reminiscent of the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. It is important to note that due to the stochastic term $D(x)\mathrm{d}w$ capturing probabilistic variations in the information transfer between the agents, the second assumption requires that the scaled net information flow $(x_i - x_j)[\sigma_{ij}(x_j) - \sigma_{ji}(x_i)]$ is bounded by the negative intensity of the diffusion coefficient given by $\frac{1}{2}\mathrm{tr}\, D(x)D^{\mathrm{T}}(x)$.

To show that (2.24) and (2.25) is stochastically semistable, note that $\mathcal{E} \triangleq f^{-1}(0) \cap D^{-1}(0) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2 = \alpha, \alpha \in \mathbb{R}\}$ and consider the Lyapunov function candidate $V(x_1, x_2) = \frac{1}{2}(x_1 - \alpha)^2 + \frac{1}{2}(x_2 - \alpha)^2$, where $\alpha \in \mathbb{R}$. Now, it follows that

$$\begin{aligned} \mathcal{L}V(x_1, x_2) &= (x_1 - \alpha)[\sigma_{12}(x_2) - \sigma_{21}(x_1)] + (x_2 - \alpha)[\sigma_{21}(x_1) - \sigma_{12}(x_2)] \\ &\quad + \tfrac{1}{2}[(\gamma(x_2 - x_1))^2 + (\gamma(x_1 - x_2))^2] \\ &= x_1[\sigma_{12}(x_2) - \sigma_{21}(x_1)] + x_2[\sigma_{21}(x_1) - \sigma_{12}(x_2)] + (\gamma(x_1 - x_2))^2 \\ &= (x_1 - x_2)[\sigma_{12}(x_2) - \sigma_{21}(x_1) + \gamma^2(x_1 - x_2)] \\ &\leq 0, \quad (x_1, x_2) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{2.26}$$

which implies that $x_1 = x_2 = \alpha$ is Lyapunov stable in probability.

Next, it is easy to see that $\mathcal{L}V(x_1, x_2) \neq 0$ when $x_1 \neq x_2$, and hence, $\mathcal{L}V(x_1, x_2) < 0$, $(x_1, x_2) \in \mathbb{R}^2 \backslash \mathcal{E}$. Therefore, it follows from Theorem 2.1 that $x_1 = x_2 = \alpha$ is stochastically semistable for all $\alpha \in \mathbb{R}$. Furthermore, $x(t) \xrightarrow{\text{a.s.}} \frac{1}{2}\mathbf{e}_2\mathbf{e}_2^{\mathrm{T}} x(0) \overset{\text{a.s.}}{=} \frac{1}{2}[x_1(0) + x_2(0)]\mathbf{e}_2$ as $t \to \infty$. Note that an identical assertion holds for the collective dynamics of n agents with a connected undirected communication graph topology. △

Finally, we provide a converse Lyapunov theorem for stochastic semistability. For this result, recall that $\mathcal{L}V(x_{\mathrm{e}}) = 0$ for every $x_{\mathrm{e}} \in \mathcal{E}$. Also note that it follows from (2.7) that

$\mathcal{L}V(x) = \mathcal{L}V(s(0,x))$.

Theorem 2.3. Consider the nonlinear stochastic dynamical system (2.2). Suppose (2.2) is stochastically semistable with a ρ-domain of semistability $\mathcal{D}_0$. Then there exist a continuous nonnegative function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ and a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$ such that $i)$ $V(x) = 0$, $x \in \mathcal{E}$, $ii)$ $V(x) \geq \alpha(\mathrm{dist}(x, \mathcal{E}))$, $x \in \mathcal{D}_0$, and $iii)$ $\mathcal{L}V(x) < 0$, $x \in \mathcal{D}_0 \backslash \mathcal{E}$.

Proof. Let $\mathfrak{B}^{x_0}$ denote the set of all sample trajectories of (2.2) for which $\lim_{t\to\infty} \mathrm{dist}(x(t, \omega), \mathcal{E}) = 0$ and $x(\{t \geq 0\}, \omega) \in \mathfrak{B}^{x_0}$, $\omega \in \Omega$, and let $\mathbb{1}_{\mathfrak{B}^{x_0}}(\omega)$, $\omega \in \Omega$, denote the indicator function defined on the set $\mathfrak{B}^{x_0}$, that is,

$$\mathbb{1}_{\mathfrak{B}^{x_0}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \geq 0\}, \omega) \in \mathfrak{B}^{x_0}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that by definition $\mathbb{P}^{x_0}(\mathfrak{B}^{x_0}) \geq 1 - \rho$ for all $x_0 \in \mathcal{D}_0$. Define the function $V : \mathcal{D}_0 \to \overline{\mathbb{R}}_+$ by

$$V(x) \triangleq \sup_{t \geq 0} \left\{ \frac{1+2t}{1+t} \mathbb{E}\left[\mathrm{dist}(s(t,x), \mathcal{E}) \mathbb{1}_{\mathfrak{B}^{x}}(\omega)\right] \right\}, \quad x \in \mathcal{D}_0, \tag{2.27}$$

and note that $V(\cdot)$ is well defined since (2.2) is stochastically semistable. Clearly, $i)$ holds. Furthermore, since $V(x) \geq \mathrm{dist}(x, \mathcal{E})$, $x \in \mathcal{D}_0$, it follows that $ii)$ holds with $\alpha(r) = r$.

To show that $V(\cdot)$ is continuous on $\mathcal{D}_0 \backslash \mathcal{E}$, define $T : \mathcal{D}_0 \backslash \mathcal{E} \to [0, \infty)$ by $T(z) \triangleq \inf\{h : \mathbb{E}\left[\mathrm{dist}(s(h,z), \mathcal{E}) \mathbb{1}_{\mathfrak{B}^{z}}(\omega)\right] < \mathrm{dist}(z, \mathcal{E})/2$ for all $t \geq h > 0\}$, and denote

$$\mathcal{W}_\varepsilon \triangleq \left\{ x \in \mathcal{D}_0 : \mathbb{P}^x \left(\sup_{t \geq 0} \mathrm{dist}(s(t,x), \mathcal{E}) \leq \varepsilon \right) \geq 1 - \rho \right\}. \tag{2.28}$$

Note that $\mathcal{W}_\varepsilon \supset \mathcal{E}$ is open and contains an open neighborhood of $\mathcal{E}$. Consider $z \in \mathcal{D}_0 \backslash \mathcal{E}$ and define $\lambda \triangleq \mathrm{dist}(z, \mathcal{E}) > 0$. Then it follows from stochastic semistability of (2.2) that there exists $h > 0$ such that $\mathbb{P}^z\left(s(h,z) \in \mathcal{W}_{\lambda/2}\right) \geq 1 - \rho$. Consequently, $\mathbb{P}^z\left(s(h+t,z) \in \mathcal{W}_{\lambda/2}\right) \geq 1 - \rho$ for all $t \geq 0$, and hence, it follows that $T(z)$ is well defined. Since $\mathcal{W}_{\lambda/2}$ is open, there exists a neighborhood $\mathcal{B}_\sigma(s(T(z), z)$ such that $\mathbb{P}^z\left(\mathcal{B}_\sigma(s(T(z),z)) \subset \mathcal{W}_{\lambda/2}\right) \geq 1 - \rho$. Hence, $\mathcal{N} \subset \mathcal{D}_0$ is a neighborhood of z such that $s_{T(z)}(\mathcal{H}_n^{\mathcal{N}}) \triangleq \mathcal{B}_\sigma(s(T(z), z))$.

Next, choose $\eta > 0$ such that $\eta < \lambda/2$ and $\mathcal{B}_\eta(z) \subset \mathcal{N}$. Then, for every $t > T(z)$ and $y \in \mathcal{B}_\eta(z)$,

$$[(1+2t)/(1+t)]\mathbb{E}\left[\text{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \leq 2\mathbb{E}\left[\text{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right] \leq \lambda.$$

Therefore, for each $y \in \mathcal{B}_\eta(z)$,

$$\begin{aligned} V(z) - V(y) &= \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\text{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right]\right\} \\ &\quad - \sup_{t\geq 0}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\text{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\right\} \\ &= \sup_{0\leq t\leq T(z)}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\text{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right]\right\} \\ &\quad - \sup_{0\leq t\leq T(z)}\left\{\frac{1+2t}{1+t}\mathbb{E}\left[\text{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\right\}. \end{aligned} \tag{2.29}$$

Hence,

$$\begin{aligned} &|V(z) - V(y)| \\ &\leq \sup_{0\leq t\leq T(z)}\left|\frac{1+2t}{1+t}\left(\mathbb{E}\left[\text{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right] - \mathbb{E}\left[\text{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\right)\right| \\ &\leq 2\sup_{0\leq t\leq T(z)}\left|\mathbb{E}\left[\text{dist}(s(t,z),\mathcal{E})\mathbb{1}_{\mathfrak{B}^z}(\omega)\right] - \mathbb{E}\left[\text{dist}(s(t,y),\mathcal{E})\mathbb{1}_{\mathfrak{B}^y}(\omega)\right]\right| \\ &\leq 2\sup_{0\leq t\leq T(z)}\mathbb{E}\left[\text{dist}(s(t,z),s(t,y))\right],\ z \in \mathcal{D}_0\backslash\mathcal{E},\ y \in \mathcal{B}_\eta(z). \end{aligned} \tag{2.30}$$

Now, since $f(\cdot)$ and $D(\cdot)$ satisfy (2.4) and (2.5), it follows from continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions [6, Thm. 7.3.1] and (2.30) that $V(\cdot)$ is continuous on $\mathcal{D}_0\backslash\mathcal{E}$.

To show that $V(\cdot)$ is continuous on $\mathcal{E}$, consider $x_\text{e} \in \mathcal{E}$. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{D}_0\backslash\mathcal{E}$ that converges to x_e. Since x_e is Lyapunov stable in probability, it follows that $x(t) \overset{\text{a.s.}}{\equiv} x_\text{e}$ is the unique solution to (2.2) with $x(0) \overset{\text{a.s.}}{=} x_\text{e}$. By continuous dependence of solutions $s(\cdot,\cdot)$ on system initial conditions [6, Thm. 7.3.1], $s(t,x_n) \overset{\text{a.s.}}{\to} s(t,x_\text{e}) \overset{\text{a.s.}}{=} x_\text{e}$ as $n \to \infty$, $t \geq 0$.

Let $\varepsilon > 0$ and note that it follows from *ii*) of Proposition 2.2 that there exists $\delta = \delta(x_\text{e}) > 0$ such that for every solution of (2.2) in $\mathcal{B}_\delta(x_\text{e})$ there exists $\hat{T} = \hat{T}(x_\text{e},\varepsilon) > 0$ such that $\mathbb{P}\left(s_t(\mathcal{H}_n^{\mathcal{B}_\delta(x_{\rm e})}) \subset \mathcal{W}_\varepsilon\right) \geq 1-\rho$ for all $t \geq \hat{T}$. Next, note that there exists a positive integer N_1 such that $x_n \in \mathcal{B}_\delta(x_{\rm e})$ for all $n \geq N_1$. Now, it follows from (2.27) that

$$V(x_n) \leq 2 \sup_{0\leq t\leq \hat{T}} \mathbb{E}[\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)] + 2\varepsilon, \quad n \geq N_1. \tag{2.31}$$

Next, it follows from [6, Thm. 7.3.1] that $\mathbb{E}[|s(\cdot,x_n)|]$ converges to $\mathbb{E}[|s(\cdot,x_{\rm e})|]$ uniformly on $[0,\hat{T}]$. Hence,

$$\begin{aligned}\lim_{n\to\infty}\sup_{0\leq t\leq \hat{T}} \mathbb{E}\left[\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right] &= \sup_{0\leq t\leq \hat{T}} \mathbb{E}\left[\lim_{n\to\infty}\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right]\\ &\leq \sup_{0\leq t\leq \hat{T}} \mathrm{dist}(x_{\rm e},\mathcal{E})\\ &= 0,\end{aligned} \tag{2.32}$$

which implies that there exists a positive integer $N_2 = N_2(x_{\rm e},\varepsilon) \geq N_1$ such that

$$\sup_{0\leq t\leq \hat{T}} \mathbb{E}\left[\mathrm{dist}(s(t,x_n),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x_n}}(\omega)\right] < \varepsilon$$

for all $n \geq N_2$. Combining (2.31) with the above result yields $V(x_n) < 4\varepsilon$ for all $n \geq N_2$, which implies that $\lim_{n\to\infty} V(x_n) = 0 = V(x_{\rm e})$.

Finally, we show that $\mathcal{L}V(x(t))$ is negative along the solution of (2.2) on $\mathcal{D}_0\backslash\mathcal{E}$. Note that for every $x \in \mathcal{D}_0\backslash\mathcal{E}$ and $0 < h \leq 1/2$ such that $\mathbb{P}\left(s(h,x) \in \mathcal{D}_0\backslash\mathcal{E}\right) \geq 1-\rho$, it follows from the definition of $T(\cdot)$ that $\mathbb{E}\left[V(s(h,x))\right]$ is reached at some time $\hat{t}$ such that $0 \leq \hat{t} \leq T(x)$. Hence, it follows from law of iterated expectation that

$$\begin{aligned}\mathbb{E}\left[V(s(h,x))\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathrm{dist}(s(\hat{t}+h,x),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{s(h,x)}}(\omega)\right]\frac{1+2\hat{t}}{1+\hat{t}}\right]\\ &= \mathbb{E}\left[\mathrm{dist}(s(\hat{t}+h,x),\mathcal{E})\mathbb{1}_{\mathfrak{B}^{x}}(\omega)\right]\frac{1+2\hat{t}+2h}{1+\hat{t}+h}\left[1-\frac{h}{(1+2\hat{t}+2h)(1+\hat{t})}\right]\\ &\leq V(x)\left[1-\frac{h}{2(1+T(x))^2}\right],\end{aligned} \tag{2.33}$$

which implies that

$$\mathcal{L}V(x) = \lim_{h\to 0^+}\frac{\mathbb{E}\left[V(s(h,x))\right]-V(x)}{h} \leq -\frac{1}{2}V(x)(1+T(x))^{-2} < 0, \quad x \in \mathcal{D}_0\backslash\mathcal{E},$$

and hence, *iii*) holds. □

Chapter 3

Nonlinear-Nonquadratic Optimal and Inverse Optimal Control for Stochastic Dynamical Systems

3.1. Introduction

Building on the results of [13, 45], in this chapter we present a framework for analyzing and designing feedback controllers for nonlinear *stochastic* dynamical systems. Specifically, we consider a feedback stochastic optimal control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional. The performance functional can be evaluated in closed form as long as the nonlinear-nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability in probability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state stochastic Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control for stochastic dynamical systems to nonlinear-nonquadratic problems.

More specifically, in Section 3.2 we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees asymptotic stability in probability. This result is then specialized to general polynomial and multilinear cost functionals. Then, in Section 3.3, we state a nonlinear-nonquadratic stochastic optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing asymptotic stability in probability of the closed-loop system. In Section 3.4, we develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the stochastic stabilization problem. This result is then used to derive extensions of the results in [11,100] involving nonlinear feedback controllers minimizing polynomial and multilinear performance criteria. Finally, in Section 3.5, we provide illustrative numerical examples that highlight the stochastic optimal stabilization framework.

3.2. Stability Analysis and Nonlinear-Nonquadratic Cost Evaluation of Nonlinear Stochastic Systems

In this section, we provide connections between Lyapunov functions and nonquadratic cost evaluation. First, we provide sufficient conditions for local and global asymptotic and exponential stability in probability for the nonlinear stochastic dynamical system (2.2). Here we assume that $f(0) = 0$ and $D(0) = 0$, and hence, $x_{\rm e} = 0$ is an equilibrium point of (2.2).

Theorem 3.1 [67, Thm. 5.3, Corol. 5.1, and Thm. 5.11]. Consider the nonlinear stochastic dynamical system (2.2) and assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0, \tag{3.1}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{3.2}$$

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2}D(x) \leq 0, \qquad x \in \mathcal{D}. \tag{3.3}$$

Then the zero solution $x(t) \equiv 0$ to (2.2) is Lyapunov stable in probability. If, in addition,

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2}D(x) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{3.4}$$

then the zero solution $x(t) \equiv 0$ to (2.2) is asymptotically stable in probability. Moreover, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to (2.2) is globally asymptotically stable in probability. Finally, if there exist scalars $\alpha, \beta, \gamma > 0$, and $p \geq 1$, such that $V : \mathcal{D} \to \mathbb{R}$ satisfies

$$\alpha\|x\|^p \le V(x) \le \beta\|x\|^p, \qquad x \in \mathcal{D}, \tag{3.5}$$

$$\frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\text{tr}\ D^{\mathrm{T}}(x)\frac{\partial^2 V(x)}{\partial x^2}D(x) \le -\gamma\|x\|^p, \qquad x \in \mathcal{D}, \tag{3.6}$$

then the zero solution $x(t) \equiv 0$ to (2.2) is exponentially p-stable in probability. Moreover, if $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to (2.2) is globally exponentially p-stable in probability.

Remark 3.1. A more general stochastic stability notion can also be introduced here involving stochastic stability and convergence to an invariant (stationary) distribution. In this case, state convergence is not to an equilibrium point but rather to a stationary distribution. This framework can relax the vanishing perturbation assumption $D(0) = 0$ and requires a more involved analysis and synthesis framework showing stability of the underlying Markov semigroup [78].

Next, we provide connections between Lyapunov functions and nonlinear-nonquadratic cost evaluation. Specifically, we present sufficient conditions for stability and performance for a given nonlinear stochastic dynamical system with a nonlinear-nonquadratic performance measure. As in deterministic theory [14, 45], the cost functional can be explicitly evaluated as long as it is related to an underlying Lyapunov function. For the following result, let $L : \mathcal{D} \to \mathbb{R}$ with $L(0) = 0$ and let $\mathbb{1}_{[t_0,\tau_m]}(t)$ denote the indicator function defined on the set $[t_0, \tau_m]$, $m \in \mathbb{Z}_+$, that is,

$$\mathbb{1}_{[t_0,\tau_m]}(t) \triangleq \begin{cases} 1, & \text{if } t \in [t_0, \tau_m], \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathfrak{B}^{\text{cost}}_{x_0}$ denote the set of all sample trajectories of the dynamical system (2.2) for which $\lim_{t\to\infty} \|x(t,\omega)\| = 0$ and $x(\{t \ge t_0\}, \omega) \in \mathfrak{B}^{\text{cost}}_{x_0}$, $\omega \in \Omega$. Finally, define

$$\mathbb{1}_{\mathfrak{B}^{\text{cost}}_{x_0}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \ge t_0\}, \omega) \in \mathfrak{B}^{\text{cost}}_{x_0}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.2. Consider the nonlinear stochastic dynamical system given by (2.2) with nonlinear-nonquadratic performance measure

$$J(x_0, \mathfrak{B}_{x_0}^{\mathrm{cost}}) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\mathrm{cost}}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}t\right]. \tag{3.7}$$

Furthermore, assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V(0) = 0, \tag{3.8}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{3.9}$$

$$V'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{3.10}$$

$$L(x) + V'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) = 0, \qquad x \in \mathcal{D}. \tag{3.11}$$

Then the zero solution $x(t) \overset{\mathrm{a.s.}}{\equiv} 0$ to (2.2) is locally asymptotically stable in probability and, for every $\rho \in (0,1)$, there exist $\delta = \delta(\rho)$ and $\mathfrak{B}_{x_0}^{\mathrm{cost}}$ with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\mathrm{cost}}\right) \geq 1-\rho$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subset \mathcal{D}$,

$$J(x_0, \mathfrak{B}_{x_0}^{\mathrm{cost}}) = V(x_0). \tag{3.12}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$ and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \overset{\mathrm{a.s.}}{\equiv} 0$ to (2.2) is globally asymptotically stable in probability and (3.12) holds with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\mathrm{cost}}\right) = 1$, $x_0 \in \mathbb{R}^n$.

Proof: Conditions (3.8)–(3.10) are a restatement of (3.1)–(3.3), and hence, it follows from Theorem 3.1 that the zero solution $x(t) \overset{\mathrm{a.s.}}{\equiv} 0$ of (2.2) is locally asymptotically stable in probability. Consequently, for every $\rho \in (0,1)$, there exist $\delta = \delta(\rho)$ and a set of sample trajectories $x(\{t \geq t_0\}, \omega) \in \mathfrak{B}_{x_0}^{\mathrm{cost}}$, $\omega \in \Omega$, such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\mathrm{cost}}\right) \geq 1-\rho$.

Next, using (2.2) and Itô's (chain rule) formula, it follows that the stochastic differential of $V(x(t))$ along the system trajectories $x(t), t \geq t_0$, of (2.2) is given by

$$\mathrm{d}V(x(t)) = \left(V'(x(t))f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))V''(x(t))D(x(t))\right)\mathrm{d}t + \frac{\partial V(x(t))}{\partial x}D(x(t))\mathrm{d}w(t). \tag{3.13}$$

Hence, using (3.11) it follows that

$$\begin{aligned} L(x(t))\mathrm{d}t + \mathrm{d}V(x(t)) &= \left(L(x(t)) + V'(x(t))f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t))V''(x(t))D(x(t))\right)\mathrm{d}t \\ &\quad + \frac{\partial V(x(t))}{\partial x}D(x(t))\mathrm{d}w(t) \\ &= \frac{\partial V(x(t))}{\partial x}D(x(t))\mathrm{d}w(t). \end{aligned} \tag{3.14}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [t_0, \infty)$ be the first exit (stopping) time of the solution $x(t)$, $t \geq t_0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, multiplying (3.14) with $\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)$ and integrating over $[t_0, \min\{t_n, \tau_m\}]$, where $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\begin{aligned} &\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}t \\ &\quad = -\int_{t_0}^{\min\{t_n,\tau_m\}} \mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}V(x(s)) + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(s))}{\partial x}D(x(s))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}w(s) \\ &\quad = V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega) \\ &\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x}D(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}w(t) \\ &\quad = V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega) \\ &\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x}D(x(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}w(t). \end{aligned} \tag{3.15}$$

Taking the expectation on both sides of (3.15) yields

$$\begin{aligned} &\mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\ \mathrm{d}t\right] \\ &\quad = \mathbb{E}^{x_0}\Bigg[V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega) \\ &\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x}D(x(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\mathrm{d}w(t)\Bigg] \\ &\quad = V(x_0)\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\mathrm{cost}}\right) - \mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{\mathrm{cost}}}(\omega)\right]. \end{aligned} \tag{3.16}$$

Next, let $\mathfrak{B}_{x_0}^m$ denote the set of all the sample trajectories $x(t)$, $t \geq t_0$, of (2.2) such that $\tau_m = \infty$ and note that, by regularity of solutions [67, p. 75], $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^m) \to 1$ as $m \to \infty$.

Now, noting that $L(x) \geq 0$, $x \in \mathcal{D}$, the sequence of random variables $\{f_{m,n}\}_{m,n=0}^{\infty} \subseteq \mathcal{H}_1$, where

$$f_{m,n} \triangleq \int_{t_0}^{\min\{t_n,\tau_m\})} L(x(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) \mathrm{d}t,$$

is a pointwise nondecreasing sequence in n and m of nonnegative $\mathcal{F}_t$-measurable random variables on Ω. Next, defining the improper integral

$$\int_{t_0}^{\infty} L(x(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) \mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from the Lebesgue monotone convergence theorem [3] that

$$\begin{aligned}
\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\ \mathrm{d}t\right] \\
&= \lim_{m\to\infty} \mathbb{E}^{x_0}\left[\lim_{n\to\infty} \int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\ \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty} \int_{t_0}^{\tau_m} L(x(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\ \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega) \mathrm{d}t\right] \\
&= J(x_0, \mathfrak{B}_{x_0}^{\text{cost}}) \mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\text{cost}}\right). \qquad (3.17)
\end{aligned}$$

Next, since the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of (2.2) is asymptotically stable in probability and $V(x(\min\{t_n, \tau_m\}))$ is a positive supermartingale by [67, Lemma 5.4], it follows from [67, Theorem 5.1] that

$$\begin{aligned}
\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\right] &= \lim_{m\to\infty} \mathbb{E}^{x_0}\left[\lim_{n\to\infty} V(x(\min\{t_n,\tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\right] \\
&= \lim_{m\to\infty} \mathbb{E}^{x_0}\left[V(x(\tau_m)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty} V(x(\tau_m)) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\right] \\
&= \mathbb{E}^{x_0}\left[V\left(\lim_{m\to\infty} x(\tau_m)\right) \mathbb{1}_{\mathfrak{B}_{x_0}^{\text{cost}}}(\omega)\right] \\
&= 0. \qquad (3.18)
\end{aligned}$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (3.16) and using (3.17) and (3.18) yields (3.12).

Finally, for $\mathcal{D} = \mathbb{R}^n$ global asymptotic stability in probability is direct consequence of the radially unbounded condition on $V(\cdot)$, and hence, (3.12) holds with $\mathbb{P}^{x_0}\left(\mathfrak{B}^{\text{cost}}_{x_0}\right) = 1$ for all $x_0 \in \mathbb{R}^n$. □

Remark 3.2. Note that for global asymptotic stability in probability $\mathbb{P}^{x_0}\left(\mathfrak{B}^{\text{cost}}_{x_0}\right) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\mathbb{1}_{\mathfrak{B}^{\text{cost}}_{x_0}}(\omega) \overset{\text{a.s.}}{=} 1$. In this case,

$$J(x_0, \mathfrak{B}^{\text{cost}}_{x_0}) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}^{\text{cost}}_{x_0}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t))\mathbb{1}_{\mathfrak{B}^{\text{cost}}_{x_0}}(\omega)\mathrm{d}t\right] = \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t))\mathrm{d}t\right].$$

Thus, in the remainder of this section we omit the dependence on $\mathfrak{B}^{\text{cost}}_{x_0}$ in the cost functional for all the results concerning global asymptotic stability in probability.

It is important to note that if (3.11) holds, then (3.10) is equivalent to $L(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$. Next, we specialize Theorem 3.2 to linear stochastic systems. For this result, let $A \in \mathbb{R}^{n\times n}$, let $\sigma \in \mathbb{R}^d$, and let $R \in \mathbb{R}^{n\times n}$ be a positive-definite matrix.

Corollary 3.1. Consider the linear stochastic dynamical system with multiplicative noise given by

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) = x_0 \quad \text{a.s.}, \quad t \geq 0, \tag{3.19}$$

and with quadratic performance measure

$$J(x_0) \triangleq \mathbb{E}^{x_0}\left[\int_0^{\infty} x^{\mathrm{T}}(t)Rx(t)\mathrm{d}t\right]. \tag{3.20}$$

Furthermore, assume that there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R. \tag{3.21}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (3.19) is globally asymptotically stable in probability and

$$J(x_0) = x_0^{\mathrm{T}} P x_0, \qquad x_0 \in \mathbb{R}^n. \tag{3.22}$$

Proof. The result is a direct consequence of Theorem 3.2 with $f(x) = Ax$, $D(x) = x\sigma^{\mathrm{T}}$, $L(x) = x^{\mathrm{T}}Rx$, $V(x) = x^{\mathrm{T}}Px$, and $\mathcal{D} = \mathbb{R}^n$. Specifically, conditions (3.8) and (3.9) are trivially satisfied. Now,

$$\begin{aligned}V'(x)f(x) + \frac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x)V''(x)D(x) &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}2P(x\sigma^{\mathrm{T}}) \\ &= x^{\mathrm{T}}\left[\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)\right]x,\end{aligned}$$

and hence, it follows from (3.21) that $L(x)+V'(x)f(x)+\frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)V''(x)D(x) = 0$, $x \in \mathbb{R}^n$, so that all the conditions of Theorem 3.2 are satisfied. Finally, since $V(\cdot)$ is radially unbounded, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (3.19) is globally asymptotically stable in probability. □

Next, we specialize Theorem 3.2 to linear and nonlinear stochastic systems with multilinear cost functionals. First, however, we give several definitions involving multilinear functions and a key lemma establishing the existence and uniqueness of specific multilinear forms. Define $x^{[q]} \triangleq x \otimes x \otimes \cdots \otimes x$ and $\overset{q}{\oplus} A \triangleq A \oplus A \oplus \cdots \oplus A$, where x and A appear q times and q is a positive integer. A scalar function $\psi : \mathbb{R}^n \to \mathbb{R}$ is *q-multilinear* if q is a positive integer and $\psi(x)$ is a linear combination of terms of the form $x_1^{i_1}x_2^{i_2}\cdots x_n^{i_n}$, where i_j is a nonnegative integer for $j = 1, \ldots, n$ and $i_1 + i_2 + \cdots + i_n = q$. Furthermore, a q-multilinear function $\psi(\cdot)$ is *nonnegative definite* (resp., *positive definite*) if $\psi(x) \geq 0$ for all $x \in \mathbb{R}^n$ (resp., $\psi(x) > 0$ for all nonzero $x \in \mathbb{R}^n$). Note that if q is odd, then $\psi(x)$ cannot be positive definite. If $\psi(\cdot)$ is a q-multilinear function, then $\psi(\cdot)$ can be represented by means of Kronecker products, that is, $\psi(x)$ is given by $\psi(x) = \Psi x^{[q]}$, where $\Psi \in \mathbb{R}^{1\times n^q}$. Note that every polynomial function can be written as a multilinear function; the converse, however, is not true.

The following lemma is needed for several of the main results of this and subsequent chapters.

Lemma 3.1. Let $A \in \mathbb{R}^{n\times n}$ and $\sigma \in \mathbb{R}^d$ be such that $A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n$ is Hurwitz, and let $h : \mathbb{R}^n \to \mathbb{R}$ be a q-multilinear function. Then there exists a unique q-multilinear function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)Ax + h(x), \qquad x \in \mathbb{R}^n. \tag{3.23}$$

Furthermore, if $h(x)$ is nonnegative (resp., positive) definite, then $g(x)$ is nonnegative (resp., positive) definite.

Proof. Let $h(x) = \Psi x^{[q]}$ and define $g(x) \triangleq \Gamma x^{[q]}$, where $\Gamma \triangleq -\Psi(\overset{q}{\oplus}\ (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n))^{-1}$, and note that $\overset{q}{\oplus}\ (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)$ is invertible since $A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n$ is Hurwitz by assumption. Now, note that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned}
&g'(x)Ax + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) \\
&\quad = (\Gamma x^{[q]})' Ax + \frac{1}{2} x^{\mathrm{T}} (\Gamma x^{[q]})'' x \|\sigma\|^2 \\
&\quad = \Gamma\left(\sum_{i_q=1}^{q} x \otimes \cdots \otimes \overbrace{I_n}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes x\right) Ax + \frac{1}{2}\|\sigma\|^2 \Bigg(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{i_q=1}^{q}\sum_{j_q=1, j_q\neq i_q}^{q} x_i \Gamma(x \otimes \cdots \\
&\qquad\qquad \cdots \otimes \overbrace{e_i}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes \overbrace{e_j}^{j_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes x) x_j\Bigg) \\
&\quad = \Gamma\left(\sum_{i_q=1}^{q} x \otimes \cdots \otimes \overbrace{Ax}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes x\right) + \frac{1}{2}\|\sigma\|^2 \Bigg(\sum_{i_q=1}^{q}\sum_{j_q=1, j_q\neq i_q}^{q}\sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma(x \otimes \cdots \\
&\qquad\qquad \cdots \otimes \overbrace{x_i e_i}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes \overbrace{x_j e_j}^{j_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes x)\Bigg) \\
&\quad = \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{A}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes I_n\right) x^{[q]} + \frac{1}{2}\|\sigma\|^2 \Bigg(\sum_{i_q=1}^{q}\sum_{j_q=1, j_q\neq i_q}^{q} \Gamma(x \otimes \cdots \\
&\qquad\qquad \cdots \otimes \overbrace{(\sum_{i=1}^{n} x_i e_i)}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes \overbrace{(\sum_{j=1}^{n} x_j e_j)}^{j_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes x)\Bigg) \\
&\quad = \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{A}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes I_n\right) x^{[q]} + \frac{1}{2}\|\sigma\|^2 \Gamma\Bigg(\sum_{i_q=1}^{q}\sum_{j_q=1, j_q\neq i_q}^{q} x \otimes \cdots \\
&\qquad\qquad \cdots \otimes \overbrace{x}^{i_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes \overbrace{x}^{j_q^{\mathrm{th}}\text{ entry}} \otimes \cdots \otimes x\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{A}^{i_q^{\text{th}} \text{ entry}} \otimes \cdots \otimes I_n\right) x^{[q]} \\
&\quad + \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{\frac{1}{2}(q-1)\|\sigma\|^2 I_n}^{i_q^{\text{th}} \text{ entry}} \otimes \cdots \otimes I_n\right) x^{[q]} \\
&= \Gamma\left(\sum_{i_q=1}^{q} I_n \otimes \cdots \otimes \overbrace{(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)}^{i_q^{\text{th}} \text{ entry}} \otimes \cdots \otimes I_n\right) x^{[q]} \\
&= \Gamma\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) x^{[q]} \\
&= -\Psi x^{[q]} \\
&= -h(x).
\end{aligned}
$$

To prove uniqueness, suppose, *ad absurdum*, that $\hat{g}(x) = \hat{\Gamma} x^{[q]}$ satisfies (3.23). Then it follows that

$$
\Gamma\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) x^{[q]} = \hat{\Gamma}\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) x^{[q]}.
$$

Since $\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)$ is Hurwitz and $e^{(\overset{q}{\oplus}(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n))t} = (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t})^{[q]}$, it follows that, for all $x \in \mathbb{R}^n$,

$$
\begin{aligned}
\Gamma x^{[q]} &= \Gamma\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right)\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right)^{-1} x^{[q]} \\
&= -\Gamma\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) \int_0^\infty e^{(\overset{q}{\oplus}(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n))t} x^{[q]} \mathrm{d}t \\
&= -\Gamma\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) \int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t})^{[q]} x^{[q]} \mathrm{d}t \\
&= -\Gamma\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) \int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x)^{[q]} \mathrm{d}t \\
&= -\hat{\Gamma}\left(\overset{q}{\oplus} (A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right) \int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x)^{[q]} \mathrm{d}t \\
&= \hat{\Gamma} x^{[q]},
\end{aligned}
$$

which shows that $g(x) = \hat{g}(x)$, $x \in \mathbb{R}^n$, leading to a contradiction.

Finally, if $h(x)$ is nonnegative definite, then it follows that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} g(x) &= -\Psi\left(\overset{q}{\oplus}\,(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n)\right)^{-1} x^{[q]} \\ &= \Psi \int_0^\infty e^{(\overset{q}{\oplus}(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n))t} x^{[q]} \mathrm{d}t \\ &= \Psi \int_0^\infty (e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x)^{[q]} \mathrm{d}t \\ &\geq 0. \end{aligned}$$

If, in addition, $x \neq 0$, then $e^{(A+\frac{1}{2}(q-1)\|\sigma\|^2 I_n)t} x \neq 0$, $t \geq 0$. Hence, if $h(x)$ is positive definite, then $g(x)$, $x \in \mathbb{R}^n$, is positive definite. □

Next, assume that $\left(A + \frac{1}{2}(q-1)\|\sigma\|^2 I_n\right)$ is Hurwitz, where $q \geq 2$ is a given integer, let P be given by (3.21), and consider the case in which $D(\cdot)$ $L(\cdot)$, $f(\cdot)$, and $V(\cdot)$ are given by $D(x) = x\sigma^{\mathrm{T}}$,

$$L(x) = x^{\mathrm{T}} R x + h(x), \quad f(x) = Ax + N(x), \quad V(x) = x^{\mathrm{T}} P x + g(x), \tag{3.24}$$

where $h : \mathcal{D} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$ are nonlinear and nonquadratic, and $N : \mathcal{D} \to \mathbb{R}^n$ is nonlinear. In this case, (3.11) holds if and only if

$$\begin{aligned} 0 = {} & x^{\mathrm{T}} R x + h(x) + x^{\mathrm{T}}(A^{\mathrm{T}} P + PA)x + 2x^{\mathrm{T}} P N(x) + g'(x)(Ax + N(x)) \\ & + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} \left[2P + g''(x)\right](x\sigma^{\mathrm{T}}), \ x \in \mathcal{D}, \end{aligned} \tag{3.25}$$

or, equivalently,

$$\begin{aligned} 0 = {} & x^{\mathrm{T}} \left[\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R\right] x \\ & + \frac{1}{2}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)(Ax + N(x)) + h(x) + 2x^{\mathrm{T}} P N(x), \ x \in \mathcal{D}. \end{aligned} \tag{3.26}$$

Since $\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)$ is Hurwitz, we can choose P to satisfy (3.21). Now, suppose $N(x) \equiv 0$ and let P satisfy (3.21). Then (3.26) specializes to

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)Ax + h(x), \quad x \in \mathcal{D}. \tag{3.27}$$

Next, given $h(\cdot)$, we determine the existence of a function $g(\cdot)$ satisfying (3.27). Here, we focus our attention on multilinear functionals for which (3.27) holds with $\mathcal{D} = \mathbb{R}^n$. Specifically, let $h(x)$ be a nonnegative-definite q-multilinear function, where q is necessarily even. Furthermore, let $g(x)$ be the nonnegative-definite q-multilinear function given by Lemma 3.1. Then, since $\frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}g''(x)(x\sigma^{\mathrm{T}}) + g'(x)Ax \leq 0$, $x \in \mathbb{R}^n$, it follows that $x^{\mathrm{T}}Px + g(x)$ is a Lyapunov function for (3.19). Hence, Lemma 3.1 can be used to generate Lyapunov functions of specific multilinear structures.

To demonstrate the above discussion suppose $h(x)$ in (3.24) is of the more general form given by

$$h(x) = \sum_{\nu=2}^{r} h_{2\nu}(x), \tag{3.28}$$

where, for $\nu = 2, 3, \ldots, r$, $h_{2\nu} : \mathbb{R}^n \to \mathbb{R}$ is a nonnegative-definite 2ν-multilinear function. Now, using Lemma 3.1, it follows that there exists a nonnegative-definite 2ν-multilinear function $g_{2\nu} : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}g''_{2\nu}(x)(x\sigma^{\mathrm{T}}) + g'_{2\nu}(x)Ax + h_{2\nu}(x), \quad x \in \mathbb{R}^n, \quad \nu = 2, 3, \ldots, r. \tag{3.29}$$

Defining $g(x) \triangleq \sum_{\nu=2}^{r} g_{2\nu}(x)$ and summing (3.29) over ν yields (3.27). Since (3.11) is satisfied with $L(x)$ and $V(x)$ given by (3.24), respectively, (3.12) implies that

$$J(x_0) = x_0^{\mathrm{T}}Px_0 + g(x_0). \tag{3.30}$$

To illustrate condition (3.27) with quartic Lyapunov functions let

$$V(x) = x^{\mathrm{T}}Px + (x^{\mathrm{T}}Mx)^2, \tag{3.31}$$

where P satisfies (3.21) and assume M is an $n \times n$ symmetric matrix. In this case, $g(x) = (x^{\mathrm{T}}Mx)^2$ is a nonnegative-definite 4-multilinear function and (3.27) yields

$$\begin{aligned} h(x) &= -4(x^{\mathrm{T}}Mx)x^{\mathrm{T}}MAx - \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}\left[8Mxx^{\mathrm{T}}M + 4(x^{\mathrm{T}}Mx)M\right](x\sigma^{\mathrm{T}}) \\ &= -2(x^{\mathrm{T}}Mx)x^{\mathrm{T}}\left[\left(A + \frac{3}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} M + M\left(A + \frac{3}{2}\|\sigma\|^2 I_n\right)\right]x. \end{aligned} \tag{3.32}$$

Now, letting M satisfy

$$0 = \left(A + \frac{3}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} M + M\left(A + \frac{3}{2}\|\sigma\|^2 I_n\right) + \hat{R}, \tag{3.33}$$

where $\hat{R}$ is an $n \times n$ symmetric matrix, it follows from (3.32) that $h(x)$ satisfying (3.27) is of the form

$$h(x) = 2(x^{\mathrm{T}} M x)(x^{\mathrm{T}} \hat{R} x). \tag{3.34}$$

If $\hat{R}$ is nonnegative definite, then M is nonnegative definite, and hence, $h(x)$ is a nonnegative-definite 4-multilinear function. Thus, if $V(x)$ is a quartic Lyapunov function of the form given by (3.31), and $L(x)$ is given by

$$L(x) = x^{\mathrm{T}} R x + 2(x^{\mathrm{T}} M x)(x^{\mathrm{T}} \hat{R} x), \tag{3.35}$$

where M satisfies (3.33), then condition (3.27), and hence, (3.11) is satisfied.

The following proposition generalizes the above results to general polynomial cost functionals.

Proposition 3.1. Let $A \in \mathbb{R}^{n\times n}$ and $\sigma \in \mathbb{R}^d$ be such that $A + \frac{1}{2}(2r-1)\|\sigma\|^2 I_n$ is Hurwitz, and let $R \in \mathbb{R}^{n\times n}$, $R > 0$, and $\hat{R}_q \in \mathbb{R}^{n\times n}$, $\hat{R}_q \geq 0$, $q = 2, \ldots, r$. Consider the linear stochastic dynamical system (3.19) with performance measure

$$J(x_0) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty \left\{x^{\mathrm{T}}(t) R x(t) + \sum_{q=2}^{r} \left[(x^{\mathrm{T}}(t)\hat{R}_q x(t))(x^{\mathrm{T}}(t) M_q x(t))^{q-1}\right]\right\} \mathrm{d}t\right], \tag{3.36}$$

where $M_q \in \mathbb{R}^{n\times n}$, and $M_q \geq 0$, $q = 2, \ldots, r$, satisfy

$$0 = \left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right)^{\mathrm{T}} M_q + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right) + \hat{R}_q. \tag{3.37}$$

Then there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R \tag{3.38}$$

and the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (3.19) is globally asymptotically stable in probability and

$$J(x_0) = x_0^{\mathrm{T}} P x_0 + \sum_{q=2}^{r} \tfrac{1}{q}(x_0^{\mathrm{T}} M_q x_0)^q, \qquad x_0 \in \mathbb{R}^n. \tag{3.39}$$

Proof. The existence of a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ for some $R > 0$ follows from converse Lyapunov theory using the fact that $A + \frac{1}{2}\|\sigma\|^2 I_n$ is Hurwitz. The result now is a direct consequence of Theorem 3.2 with $f(x) = Ax$, $D(x) = x\sigma^{\mathrm{T}}$, $L(x) = x^{\mathrm{T}} R x + \sum_{q=2}^{r}[(x^{\mathrm{T}}\hat{R}_q x)(x^{\mathrm{T}} M_q x)^{q-1}]$, $V(x) = x^{\mathrm{T}} P x + \sum_{q=2}^{r} \frac{1}{q}(x^{\mathrm{T}} \ M_q x)^q$, and $\mathcal{D} = \mathbb{R}^n$. Specifically, conditions (3.8) and (3.9) are trivially satisfied. Now,

$$\begin{aligned} V'(x)f(x) \ + \ & \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}x^{\mathrm{T}}(A^{\mathrm{T}}M_q + M_q A)x \\ &\quad + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}\left[2P + 4(q-1)(x^{\mathrm{T}}M_q x)^{q-2}M_q x x^{\mathrm{T}} M_q + 2(x^{\mathrm{T}}M_q x)M_q\right](x\sigma^{\mathrm{T}}) \\ &= x^{\mathrm{T}}\left[\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)\right]x \\ &\quad + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}x^{\mathrm{T}}\left[\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right)^{\mathrm{T}} M_q \right. \\ &\quad \left. + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n\right)\right]x, \end{aligned}$$

and hence, it follows from (3.37) and (3.38) that $L(x) + V'(x)f(x) + \frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)V''(x)D(x) = 0$, $x \in \mathbb{R}^n$, so that all the conditions of Theorem 3.2 are satisfied. Finally, since $V(\cdot)$ is radially unbounded (3.19) is globally asymptotically stable in probability. □

Remark 3.3. Proposition 3.1 requires the solutions of $r-1$ Lyapunov equations in (3.37) to obtain a closed-form expression for the nonlinear-nonquadratic cost functional (3.36).

3.3. Stochastic Optimal Nonlinear Control

In this section, we consider a control problem involving a notion of optimality with respect to a nonlinear-nonquadratic cost functional. We use the framework developed in Theorem 3.2

to obtain a characterization of optimal feedback controllers that guarantee closed-loop local and global stabilization in probability. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the stochastic Hamilton-Jacobi-Bellman equation. To address the problem of characterizing stochastic optimal stabilizing feedback controllers, consider the controlled nonlinear stochastic dynamical system $\mathcal{G}$ given by

$$\mathrm{d}x(t) = F(x(t), u(t))\mathrm{d}t + D(x(t), u(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{3.40}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $x(0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$ is open set with $0 \in U$, $w(\cdot)$ is a d-dimensional independent standard Wiener process, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is jointly continuous in x and u with $F(0,0) = 0$, and $D : \mathcal{D} \times U \to \mathbb{R}^{n\times d}$ is jointly continuous in x and u with $D(0,0) = 0$.

Here we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that (3.40) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (3.40) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq t_0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau)$, $w(\tau)$, $\tau \leq s$, and $x(t_0)$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume that $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity and growth conditions (2.4) and (2.5) hold for the controlled drift and diffusion terms $F(x,u)$ and $D(x,u)$ uniformly in u. In this case, it follows from Theorem 2.2.4 of [5] that there exists a pathwise unique solution to (3.40) in $(\Omega, \{\mathcal{F}_{t\geq t_0}\}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq t_0$, where $\phi(\cdot)$ is a control law and $x(t)$, $t \geq t_0$, satisfies (3.40), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq t_0$, the *closed-loop system* (3.40) has the form

$$\mathrm{d}x(t) = F(x(t), \phi(x(t)))\mathrm{d}t + D(x(t), \phi(x(t)))\mathrm{d}w(t) \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0. \tag{3.41}$$

Next, we present a main theorem for stochastic stabilization characterizing feedback controllers that guarantee local and global closed-loop stability in probability and minimize a nonlinear-nonquadratic performance measure. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ be jointly continuous in x and u, and, for every $\rho \in (0,1)$, define the set of stochastic regulation controllers given by

$$\mathcal{S}(x_0,\rho) \triangleq \Big\{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (3.40) is such that } \mathbb{P}^{x_0}\Big(\mathfrak{B}_{x_0}^{u(\cdot)}\Big) \\ \geq 1-\rho, \text{ where } \mathfrak{B}_{x_0}^{u(\cdot)} \triangleq \Big\{x(\{t \geq t_0\},\omega) : \lim_{t\to\infty} \|x(t,\omega)\| = 0,\, \omega \in \Omega\Big\} \Big\}.$$

Theorem 3.3. Consider the nonlinear stochastic controlled dynamical system (3.40) with performance measure

$$J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right], \tag{3.42}$$

where $u(\cdot)$ is an admissible control and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)$ denotes the indicator function of the set $\mathfrak{B}_{x_0}^{u(\cdot)}$. Assume that there exists a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$V(0) = 0, \tag{3.43}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{3.44}$$

$$\phi(0) = 0, \tag{3.45}$$

$$V'(x)F(x,\phi(x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,\phi(x))V''(x)D(x,\phi(x)) < 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{3.46}$$

$$H(x,\phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{3.47}$$

$$H(x,u) \geq 0, \qquad x \in \mathcal{D}, \qquad u \in U, \tag{3.48}$$

where

$$H(x,u) \triangleq L(x,u) + V'(x)F(x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,u)V''(x)D(x,u). \tag{3.49}$$

Then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (3.41) is locally asymptotically stable in probability and, for every $\rho \in (0,1)$, there exist $\delta = \delta(\rho)$ and $\mathfrak{B}_{x_0}^{\phi(x(\cdot))}$ with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) \geq 1 - \rho$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$,

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = V(x_0). \tag{3.50}$$

In addition, if $x_0 \in \mathcal{B}_\delta(0)$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes (3.42) in the sense that

$$J\left(x_0, \phi(x(\cdot)), \mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = \min_{u(\cdot) \in \mathcal{S}(x_0,\rho)} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right). \tag{3.51}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $V(x) \to \infty$ as $\|x\| \to \infty$, then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (3.41) is globally asymptotically stable in probability and (3.51) holds with $\rho = 0$ and $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = 1$, $x_0 \in \mathbb{R}^n$.

Proof. Local and global asymptotic stability in probability are a direct consequence of (3.43)–(3.46) by applying Theorem 3.2 to the closed-loop system (3.41). Furthermore, using (3.47), condition (3.50) is a restatement of (3.12) as applied to the closed-loop system. Consequently, for every $\rho \in (0, 1)$, there exist $\delta = \delta(\rho)$ and a set of sample trajectories $x(\{t \geq t_0\}, \omega) \in \mathfrak{B}_{x_0}^{\phi(x(\cdot))}$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) \geq 1 - \rho$.

Next, let $x_0 \in \mathcal{B}_\delta(0)$, let $u(\cdot) \in \mathcal{S}(x_0, \rho)$, and let $x(t)$, $t \geq t_0$, be the solution of (3.40). Then using Itô's (chain rule) formula it follows that

$$\begin{aligned} L(x(t), u(t))\mathrm{d}t + \mathrm{d}V(x(t)) = &\Big(L(x(t), u(t)) + V'(x(t))F(x, u(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x(t), u(t)) \\ &\cdot V''(x(t))D(x(t), u(t))\Big)\mathrm{d}t + \frac{\partial V(x)}{\partial x}D(x, u)\mathrm{d}w(t), \end{aligned}$$

and hence,

$$L(x(t), u(t))\mathrm{d}t = -\mathrm{d}V(x(t)) + H(x(t), u(t))\mathrm{d}t + \frac{\partial V(x(t))}{\partial x}D(x(t), u(t))\mathrm{d}w(t). \tag{3.52}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [t_0, \infty)$ be the first exit (stopping) time of the solution $x(t)$, $t \geq t_0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, multiplying (3.52) with $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)$ and integrating over

$[0, \min\{t_n, \tau_m\}]$, where $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$
\begin{aligned}
&\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \, \mathrm{d}t \\
&\quad = -\int_{t_0}^{\min\{t_n,\tau_m\}} \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}V(x(t)) + \int_{t_0}^{\min\{t_n,\tau_m\}} H(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}t \\
&\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}w(t) \\
&\quad = V(x(t_0)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) - V(x(\min\{t_n, \tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \\
&\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} H(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}t \\
&\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t), u(t)) \mathbb{1}_{[t_0,\tau_m]}(t) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}w(t).
\end{aligned}
\tag{3.53}
$$

Next, taking the expectation on both sides of (3.53) and using (3.48) yields

$$
\begin{aligned}
&\mathbb{E}^{x_0}\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \, \mathrm{d}t\right] \\
&\quad = \mathbb{E}^{x_0}\Bigg[V(x(t_0)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) - V(x(\min\{t_n, \tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \\
&\qquad + \int_{t_0}^{\min\{t_n,\tau_m\}} H(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}t \\
&\qquad + \int_{t_0}^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t), u(t)) \mathbb{1}_{[t_0,\tau_m]}(t) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \mathrm{d}w(t)\Bigg] \qquad (3.54) \\
&\quad \geq V(x_0) \mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right) - \mathbb{E}^{x_0}\left[V(x(\min\{t_n, \tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right]. \qquad (3.55)
\end{aligned}
$$

Next, let $\mathfrak{B}_{x_0}^m$ denote the set of all the sample trajectories of $x(t)$, $t \geq t_0$, such that $\tau_m = \infty$ and note that, by regularity of solutions [67, p. 75], $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^m) \to 1$ as $m \to \infty$. Now, noting that for all $u(\cdot) \in \mathcal{S}(x_0, \rho)$,

$$
\int_0^\infty \left| L(x(t), u(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \right| \mathrm{d}t \overset{\text{a.s.}}{<} \infty,
$$

let the random variable

$$
g \triangleq \sup_{t \geq 0, m > 0} \int_0^{\min\{t,\tau_m\}} \left| L(x(s), u(s)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \right| \mathrm{d}s.
$$

In this case, the sequence in n and m of $\mathcal{F}_t$-measurable random variables $\{f_{m,n}\}_{m,n=0}^{\infty} \subseteq \mathcal{H}_1$ on Ω for all $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, where

$$f_{m,n} \triangleq \int_0^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t$$

satisfies $|f_{m,n}| \overset{\text{a.s.}}{<} g$, $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Now, defining the improper integral

$$\int_0^{\infty} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from the dominated convergence theorem [3] that

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}^{x_0}&\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_{t_0}^{\min\{t_n,\tau_m\}} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_{t_0}^{\tau_m} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\
&= J\left(x_0,u(\cdot),\mathfrak{B}_{x_0}^{u(\cdot)}\right)\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right). \qquad (3.56)
\end{aligned}$$

Next, using the fact that $u(\cdot) \in \mathcal{S}(x_0,\rho)$ and $V(\cdot)$ is continuous, it follows that for every $m > 0$, $V(x((\min\{t_n,\tau_m\}))$ is bounded for all $n \in \mathbb{Z}_+$. Thus, using the dominated convergence theorem [3] and the fact that $\|x(t,\omega)\| \to 0$ as $t \to \infty$ for all $x(\{t \geq t_0\},\omega) \in \mathfrak{B}_{x_0}^{u(\cdot)}$, we obtain

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right] &= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty}V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}V(x(\tau_m))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right] \\
&= \mathbb{E}^{x_0}\left[V\left(\lim_{m\to\infty}x(\tau_m)\right)\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\right] \\
&= 0. \qquad (3.57)
\end{aligned}$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (3.55) and using the fact that $u(\cdot) \in \mathcal{S}(x_0,\rho)$, (3.56), (3.57), and $J\left(x_0,\phi(x(\cdot)),\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = V(x_0)$ yields (3.51).

Finally, for $\mathcal{D} = \mathbb{R}^n$ global asymptotic stability in probability of closed-loop system is direct consequence of the radially unbounded condition on $V(\cdot)$, and hence, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot))}\right) = 1$ for all $x_0 \in \mathbb{R}^n$. In this case, the proof of (3.51) follows using identical arguments as in the proof of the local result. □

Note that (3.47) is the steady-state stochastic Hamilton-Jacobi-Bellman equation. To see this, recall that the stochastic Hamilton-Jacobi-Bellman equation is given by ([6])

$$\frac{\partial}{\partial t}V(t,x(t)) + \min_{u\in U} H\left(t, x(t), u, \frac{\partial}{\partial x}V(t,x(t)), \frac{\partial^2}{\partial x^2}V(t,x(t))\right) = 0, \quad t \geq t_0, \tag{3.58}$$

which characterizes the optimal control for stochastic time-varying systems on a finite or infinite interval. For infinite horizon time-invariant systems, $V(t,x) = V(x)$, and hence, (3.58) reduces to (3.47) and (3.48). Conditions (3.47) and (3.48) guarantee optimality with respect to the set of admissible stabilizing controllers $\mathcal{S}(x_0, \rho)$. However, it is important to note that an explicit characterization of the set $\mathcal{S}(x_0, \rho)$ is not required. In addition, the optimal stabilizing *feedback* control law $u = \phi(x)$ is independent of the initial condition x_0. Finally, in order to ensure asymptotic stability in probability of the closed-loop system (3.40), Theorem 3.3 requires that $V(\cdot)$ satisfy (3.43), (3.44), and (3.46), which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (3.40). However, for optimality $V(\cdot)$ need not satisfy (3.44) and (3.46). Specifically, if $V(\cdot)$ is a two-times continuously differentiable function such that (3.43) is satisfied and $\phi(\cdot) \in \mathcal{S}(x_0, \rho)$, then (3.47) and (3.48) imply (3.50) and (3.51).

The optimal feedback control $\phi(\cdot)$ that guarantees global asymptotic stability in probability gives $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right) = 1$, and hence, $\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. Moreover, all the admissible controls $u(\cdot)$ that guarantee global attraction in probability satisfy $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\rho = 0$ and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. In this case,

$$\begin{aligned} J\left(x_0, u(\cdot), \mathfrak{B}_{x_0}^{u(\cdot)}\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot)}}(\omega)\mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x(t),u(t))\mathrm{d}t\right] \end{aligned} \tag{3.59}$$

and

$$J\left(x_0,\phi(\cdot),\mathfrak{B}_{x_0}^{\phi(\cdot)}\right)=\frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot)}\right)}\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty}L(x(t),\phi(x(t)))\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot)}}(\omega)\mathrm{d}t\right]$$
$$=\mathbb{E}^{x_0}\left[\int_{t_0}^{\infty}L(x(t),\phi(x(t)))\mathrm{d}t\right]. \quad (3.60)$$

Thus, in the remainder of the chapter, we omit the dependence on $\mathfrak{B}_{x_0}^{\phi(\cdot)}$ and $\mathfrak{B}_{x_0}^{u(\cdot)}$ in the cost functional and we write $\mathcal{S}(x_0)$ for $\mathcal{S}(x_0,\rho)$ for all the results concerning globally stabilizing controllers in probability.

Next, we specialize Theorem 3.3 to linear stochastic dynamical systems and provide connections to the stochastic optimal linear-quadratic regulator problem with multiplicative noise. For the following result let $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $\sigma\in\mathbb{R}^{d}$, $R_1\in\mathbb{P}^{n}$, and $R_2\in\mathbb{P}^{m}$ be given.

Corollary 3.2. Consider the linear controlled stochastic dynamical system with multiplicative noise given by

$$\mathrm{d}x(t)=[Ax(t)+Bu(t)]\,\mathrm{d}t+x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t),\qquad x(0)\overset{\text{a.s.}}{=}x_0,\qquad t\geq 0, \quad (3.61)$$

and with quadratic performance measure

$$J(x_0,u(\cdot))\triangleq\mathbb{E}^{x_0}\left[\int_0^{\infty}[x^{\mathrm{T}}(t)R_1x(t)+u^{\mathrm{T}}(t)R_2u(t)]\mathrm{d}t\right], \quad (3.62)$$

where $u(\cdot)$ is an admissible control. Furthermore, assume that there exists a positive-definite matrix $P\in\mathbb{R}^{n\times n}$ such that

$$0=\left(A+\frac{1}{2}\|\sigma\|^2I_n\right)^{\mathrm{T}}P+P\left(A+\frac{1}{2}\|\sigma\|^2I_n\right)+R_1-PBR_2^{-1}B^{\mathrm{T}}P. \quad (3.63)$$

Then, with the feedback control $u=\phi(x)\triangleq -R_2^{-1}B^{\mathrm{T}}Px$, the zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ to (3.61) is globally asymptotically stable in probability and

$$J(x_0,\phi(x(\cdot)))=x_0^{\mathrm{T}}Px_0,\qquad x_0\in\mathbb{R}^n. \quad (3.64)$$

Furthermore,

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \tag{3.65}$$

where $\mathcal{S}(x_0)$ is the set of regulation controllers for (3.61) and $x_0 \in \mathbb{R}^n$.

Proof. The result is a direct consequence of Theorem 3.3 with $F(x,u) = Ax + Bu$, $D(x,u) = x\sigma^{\mathrm{T}}$, $L(x,u) = x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_2u$, $V(x) = x^{\mathrm{T}}Px$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. Specifically, conditions (3.43) and (3.44) are trivially satisfied. Next, it follows from (3.63) that $H(x,\phi(x)) = 0$, and hence, $V'(x)F(x,\phi(x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,\phi(x))V''(x)D(x,\phi(x)) < 0$ for all $x \in \mathbb{R}^n$ and $x \neq 0$. Thus, $H(x,u) = H(x,u) - H(x,\phi(x)) = [u-\phi(x)]^{\mathrm{T}}R_2[u-\phi(x)] \geq 0$ so that all the conditions of Theorem 3.3 are satisfied. Finally, since $V(\cdot)$ is radially unbounded the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (3.61), with $u(t) = \phi(x(t)) = -R_2^{-1}B^{\mathrm{T}}Px(t)$, is globally asymptotically stable in probability. □

The optimal feedback control law $\phi(x)$ in Corollary 3.2 is derived using the properties of $H(x,u)$ as defined in Theorem 3.3. Specifically, since $H(x,u) = x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_2u + x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + 2x^{\mathrm{T}}PBu + \|\sigma\|^2x^{\mathrm{T}}Px$ it follows that $\frac{\partial^2 H}{\partial u^2} = R_2 > 0$. Now, $\frac{\partial H}{\partial u} = 2R_2u + 2B^{\mathrm{T}}Px = 0$ gives the unique global minimum of $H(x,u)$. Hence, since $\phi(x)$ minimizes $H(x,u)$ it follows that $\phi(x)$ satisfies $\frac{\partial H}{\partial u} = 0$ or, equivalently, $\phi(x) = -R_2^{-1}B^{\mathrm{T}}Px$.

3.4. Inverse Optimal Stochastic Control for Nonlinear Affine Systems

In this section, we specialize Theorem 3.3 to affine in the control systems. Specifically, we construct nonlinear feedback controllers using a stochastic optimal control framework that minimizes a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller such that the mapping of the infinitesimal generator of the Lyapunov function is negative definite along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing (in probability) solutions to the stochastic Hamilton-Jacobi-Bellman equation. Thus, these results provide a family of globally stabilizing controllers parameterized by the cost functional that is minimized.

The controllers obtained in this section are predicated on an *inverse optimal stochastic control problem* [34, 42, 57, 58, 79, 82, 95]. In particular, to avoid the complexity in solving the stochastic steady-state Hamilton-Jacobi-Bellman equation we do not attempt to minimize a *given* cost functional, but rather, we parameterize a family of stochastically stabilizing controllers that minimize some *derived* cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function for the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the stochastic Hamilton-Jacobi-Bellman equation. Hence, by varying parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing in probability controllers that can meet closed-loop system response constraints.

Consider the nonlinear stochastic affine in the control dynamical system given by

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{3.66}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $D : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfies $D(0) = 0$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. Furthermore, we consider performance integrands $L(x, u)$ of the form

$$L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u, \tag{3.67}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^n \to \mathbb{P}^m$ so that (3.42) becomes

$$J(x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t)]\mathrm{d}t\right]. \tag{3.68}$$

Theorem 3.4. Consider the nonlinear controlled affine stochastic dynamical system (3.66) with performance measure (3.68). Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and a function $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ such that

$$V(0) = 0, \tag{3.69}$$

$$L_2(0) = 0, \tag{3.70}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{3.71}$$

$$V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{3.72}$$

and $V(x) \to \infty$ as $\|x\| \to \infty$. Then the zero solution $x(t) \stackrel{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))\phi(x(t))]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{3.73}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[V'(x)G(x) + L_2(x)]^{\mathrm{T}}, \tag{3.74}$$

and the performance measure (3.68), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x) - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x), \tag{3.75}$$

is minimized in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)), \qquad x_0 \in \mathbb{R}^n. \tag{3.76}$$

Finally,

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{3.77}$$

Proof. The result is a direct consequence of Theorem 3.3 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $F(x,u) = f(x) + G(x)u$, $D(x,u) = D(x)$, and $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u$. Specifically, with (3.67) the Hamiltonian has the form

$$H(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u + V'(x)(f(x) + G(x)u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x).$$

Now, the feedback control law (3.74) is obtained by setting $\frac{\partial H}{\partial u} = 0$. With (3.74), it follows that (3.69), (3.71), and (3.72) imply (3.43), (3.44), and (3.46), respectively. Next, since $V(\cdot)$ is two-times continuously differentiable and $x = 0$ is a local minimum of $V(\cdot)$, it follows that

$V'(0) = 0$, and hence, since by assumption $L_2(0) = 0$, it follows that $\phi(0) = 0$, which implies (3.45). Next, with $L_1(x)$ given by (3.75) and $\phi(x)$ given by (3.74), (3.47) holds. Finally, since $H(x,u) = H(x,u) - H(x,\phi(x)) = [u - \phi(x)]^{\mathrm{T}} R_2(x)[u - \phi(x)]$ and $R_2(x)$ is positive definite for all $x \in \mathbb{R}^n$, condition (3.48) holds. The result now follows as a direct consequence of Theorem 3.3. $\square$

Note that (3.72) is equivalent to

$$\mathcal{L}V(x) \triangleq V'(x)[f(x) + G(x)\phi(x)] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \quad (3.78)$$

with $\phi(x)$ given by (3.74). Furthermore, conditions (3.69), (3.71), and (3.78) ensure that $V(\cdot)$ is a Lyapunov function for the closed-loop system (3.73). As discussed in [45], it is important to recognize that the function $L_2(x)$, which appears in the integrand of the performance measure (3.67), is an arbitrary function of $x \in \mathbb{R}^n$ subject to conditions (3.70) and (3.72). Thus, $L_2(x)$ provides flexibility in choosing the control law.

With $L_1(x)$ given by (3.75) and $\phi(x)$ given by (3.74), $L(x,u)$ can be expressed as

$$\begin{aligned} L(x,u) &= u^{\mathrm{T}} R_2(x) u - \phi^{\mathrm{T}}(x) R_2(x)\phi(x) + L_2(x)(u - \phi(x)) \\ &\quad - V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ &= \left[u + \tfrac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right]^{\mathrm{T}} R_2(x)\left[u + \tfrac{1}{2}R_2^{-1}(x)L_2^{\mathrm{T}}(x)\right] - V'(x)[f(x) \\ &\quad + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) - \tfrac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x). \quad (3.79) \end{aligned}$$

Since $R_2(x) > 0$, $x \in \mathbb{R}^n$, the first term on the right-hand side of (3.79) is nonnegative, while (3.78) implies that the second, third, and fourth terms collectively are nonnegative. Thus, it follows that

$$L(x,u) \geq -\tfrac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \quad (3.80)$$

which shows that $L(x,u)$ may be negative. As a result, there may exist a control input u for which the performance measure $J(x_0,u)$ is negative. However, if the control u is a regulation controller, that is, $u \in \mathcal{S}(x_0)$, then it follows from (3.76) and (3.77) that

$$J(x_0, u(\cdot)) \geq V(x_0) \geq 0, \qquad x_0 \in \mathbb{R}^n, \qquad u(\cdot) \in \mathcal{S}(x_0). \quad (3.81)$$

Furthermore, in this case, substituting $u = \phi(x)$ into (3.79) yields

$$L(x, \phi(x)) = -V'(x)[f(x) + G(x)\phi(x)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x), \tag{3.82}$$

which, by (3.78), is positive.

Next, we specialize Theorem 3.4 to linear stochastic systems controlled by nonlinear controllers that minimize a polynomial cost functional. For the following result let $\sigma \in \mathbb{R}^d$, $R_1 \in \mathbb{P}^n$, $R_2 \in \mathbb{P}^m$, and $\hat{R}_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, be given, where r is a positive integer, and define $S \triangleq BR_2^{-1}B^{\mathrm{T}}$.

Corollary 3.3. Consider the linear controlled stochastic dynamical system (3.61). Assume that there exist $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, \ldots, r$, such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R_1 - PSP, \tag{3.83}$$

$$0 = \left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right)^{\mathrm{T}} M_q + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right) + \hat{R}_q, \quad q = 2, \ldots, r. \tag{3.84}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = (Ax(t) + B\phi(x(t)))\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{3.85}$$

is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}\left(P + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right)x, \tag{3.86}$$

and the performance measure (3.68) with $R_2(x) = R_2$, $L_2(x) = 0$, and

$$L_1(x) = x^{\mathrm{T}}\left(R_1 + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}\hat{R}_q + \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} S \cdot \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]\right)x, \tag{3.87}$$

is minimized in the sense of (3.76). Finally,

$$J(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}} P x_0 + \sum_{q=2}^{r} \tfrac{1}{q}(x_0^{\mathrm{T}} M_q x_0)^q, \qquad x_0 \in \mathbb{R}^n. \tag{3.88}$$

Proof. The result is a direct consequence of Theorem 3.4 with $f(x) = Ax$, $G(x) = B$, $D(x) = x\sigma^{\mathrm{T}}$, $L_2(x) = 0$, $R_2(x) = R_2$, and

$$V(x) = x^{\mathrm{T}} P x + \sum_{q=2}^{r} \frac{1}{q}(x^{\mathrm{T}} M_q x)^q.$$

Specifically, (3.69)–(3.71) are trivially satisfied. Next, it follows from (3.83), (3.84), and (3.86) that

$$\begin{aligned} V'(x)[f(x) - \tfrac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ = -x^{\mathrm{T}}R_1x - \sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}x^{\mathrm{T}}\hat{R}_qx - \phi^{\mathrm{T}}(x)R_2\phi(x) \\ -x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]^{\mathrm{T}} S \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]x, \end{aligned}$$

which implies (3.72), so that all the conditions of Theorem 3.4 are satisfied. □

Corollary 3.3 requires the solutions of $r-1$ modified Riccati equations in (3.84) to obtain the optimal controller (3.86). It is important to note that the derived performance measure weighs the state variables by arbitrary even powers. Furthermore, $J(x_0, u(\cdot))$ has the form

$$\begin{aligned} J(x_0, u(\cdot)) \;=\; \mathbb{E}^{x_0}\Bigg[\int_0^\infty \Bigg\{ x^{\mathrm{T}}(t)\left(R_1 + \sum_{q=2}^{r}(x^{\mathrm{T}}(t)M_qx(t))^{q-1}\hat{R}_q\right)x(t) + u^{\mathrm{T}}(t)R_2u(t) \\ +\phi_{\mathrm{NL}}^{\mathrm{T}}(x(t))R_2\phi_{\mathrm{NL}}(x(t))\Bigg\}\mathrm{d}t\Bigg], \end{aligned}$$

where $\phi_{\mathrm{NL}}(x)$ is the nonlinear part of the optimal feedback control

$$\phi(x) = \phi_{\mathrm{L}}(x) + \phi_{\mathrm{NL}}(x),$$

where $\phi_{\mathrm{L}}(x) \triangleq -R_2^{-1}B^{\mathrm{T}}Px$ and $\phi_{\mathrm{NL}}(x) \triangleq -R_2^{-1}B^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_qx$.

Remark 3.4. Corollary 3.3 generalizes the stochastic nonlinear-nonquadratic optimal control problem considered in [100] to polynomial performance criteria. Specifically, unlike the results of [100], Corollary 3.3 is not limited to sixth-order cost functionals and cubic nonlinear controllers but rather addresses a polynomial performance criterion of an arbitrary even order.

Remark 3.5. General nonquadratic cost functions can result in nonlinear controllers that yield a faster than exponential closed-loop system response. Alternatively, when the nonlinear-nonquadratic performance measure involves terms of order x^p, where $p < 2$, then we have a *subquadratic* cost criterion, which pays close attention to the system state near the origin. In this case, the optimal controller is *sublinear*, and hence, exhibits finite settling time behavior [48]. This is further discussed in Chapter 5.

Next, we specialize Theorem 3.4 to linear stochastic systems controlled by nonlinear controllers that minimize a multilinear cost functional. For the following result recall the definition of S and let $R_1 \in \mathbb{P}^n$, $R_2 \in \mathbb{P}^m$, and $\hat{R}_{2q} \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, r$, be given, where r is a given integer and $\mathcal{N}^{(k,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^k} : \Psi x^{[k]} \geq 0, x \in \mathbb{R}^n\}$.

Corollary 3.4. Consider the linear controlled stochastic dynamical system (3.61). Assume that there exist $P \in \mathbb{P}^n$ and $\hat{P}_q \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, r$, such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R_1 - PSP, \tag{3.89}$$

$$0 = \hat{P}_q \left[\overset{2q}{\oplus} \left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right)\right] + \hat{R}_{2q}, \qquad q = 2, \ldots, r. \tag{3.90}$$

Then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (3.85) is globally asymptotically stable in probability with the feedback control law

$$\phi(x) = -R_2^{-1} B^{\mathrm{T}}(Px + \tfrac{1}{2} g'^{\mathrm{T}}(x)), \tag{3.91}$$

where $g(x) \triangleq \sum_{q=2}^{r} \hat{P}_q x^{[2q]}$, and the performance measure (3.68) with $R_2(x) = R_2$, $L_2(x) = 0$, and

$$L_1(x) = x^{\mathrm{T}} R_1 x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} + \tfrac{1}{4} g'(x) S g'^{\mathrm{T}}(x), \qquad (3.92)$$

is minimized in the sense of (3.76). Finally,

$$J(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}} P x_0 + \sum_{q=2}^{r} \hat{P}_q x_0^{[2q]}, \qquad x_0 \in \mathbb{R}^n. \qquad (3.93)$$

Proof. The result is a direct consequence of Theorem 3.4 with $f(x) = Ax$, $G(x) = B$, $D(x) = x\sigma^{\mathrm{T}}$, $L_2(x) = 0$, $R_2(x) = R_2$, and $V(x) = x^{\mathrm{T}} P x + \sum_{q=2}^{r} \hat{P}_q x_0^{[2q]}$. Specifically, (3.69)–(3.71) are trivially satisfied. Next, it follows from (3.89)–(3.91) that

$$V'(x)[f(x) - \tfrac{1}{2} G(x) R_2^{-1}(x) G^{\mathrm{T}}(x) V'^{\mathrm{T}}(x)] + \frac{1}{2} \mathrm{tr}\ D^{\mathrm{T}}(x) V''(x) D(x)$$
$$= -x^{\mathrm{T}} R_1 x - \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]} - \phi^{\mathrm{T}}(x) R_2 \phi(x) - \tfrac{1}{4} g'(x) S g'^{\mathrm{T}}(x),$$

which implies (3.72) so that all the conditions of Theorem 3.4 are satisfied. □

Note that since

$$g'(x)(A - SP)x + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) = \sum_{q=2}^{r} \hat{P}_q \Big[\overset{2q}{\oplus} \big(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\big)\Big] x^{[2q]},$$

it follows that (3.90) can be equivalently written as

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)(A - SP)x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]}, \quad x \in \mathbb{R}^n,$$

and hence, it follows from Lemma 3.1 that there exists a unique $\hat{P}_q \in \mathcal{N}^{(2q,n)}$ such that (3.90) is satisfied.

Remark 3.6. Corollary 3.4 generalizes the deterministic nonlinear feedback controller results obtained by Bass and Webber in [11] to stochastic nonlinear feedback control.

3.5. Illustrative Numerical Examples

In this section, we present two numerical examples to demonstrate the efficacy of the proposed approach.

Example 3.1. First, we consider an academic example involving the two-state controlled nonlinear stochastic dynamical system given by

$$\mathrm{d}x_1(t) = -x_1(t)\mathrm{d}t + u_1(t)\mathrm{d}t + x_2^2(t)\mathrm{d}w(t), \qquad x_1(0) \stackrel{\text{a.s.}}{=} x_{10}, \qquad t \geq 0, \tag{3.94}$$

$$\mathrm{d}x_2(t) = -x_2^3(t)\mathrm{d}t + u_2(t)\mathrm{d}t + x_1(t)\mathrm{d}w(t), \qquad x_2(0) \stackrel{\text{a.s.}}{=} x_{20}, \tag{3.95}$$

and note that (5.97) and (5.98) can be cast in the form of (3.66) with $f(x) = [-x_1,\ -x_2^3]^{\mathrm{T}}$, $G(x) = I_2$, and $D(x) = [x_2^2,\ x_1]^{\mathrm{T}}$, where $x \triangleq [x_1\ x_2]^{\mathrm{T}}$. To construct an inverse optimal globally stabilizing control law for (5.97) and (5.98), let $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ and let $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2u$, where $R_2 > 0$. Now, $L_2(x) = x^{\mathrm{T}}$ satisfies (3.70) so that the inverse optimal control law (3.74) is given by $\phi(x) = -R_2^{-1}x$. In this case, the performance measure (3.68), with $L_1(x) = x^{\mathrm{T}}R_2^{-1}x + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^4$, is minimized in the sense of (3.76). Furthermore, since $V(x)$ is radially unbounded and

$$\mathcal{L}V(x) = -x^{\mathrm{T}}R_2^{-1}x - \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^4 < 0, \qquad x \in \mathbb{R}^2, \qquad x \neq 0, \tag{3.96}$$

the feedback control law $\phi(x) = -R_2^{-1}x$ is globally stabilizing in probability.

Let $x(0) = [1,\ -1]^{\mathrm{T}}$ a.s. and $R_2 = 4I_2$. Figure 3.1 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figure 3.2 shows the sample average along with the standard deviation of the corresponding control signal versus time for 20 sample paths. △

Example 3.2. Consider the pitch axis longitudinal dynamics model of the F-16 fighter aircraft system for nominal flight conditions at 3000 ft and Mach number of 0.6 with stochastic disturbances given by ([45])

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{3.97}$$

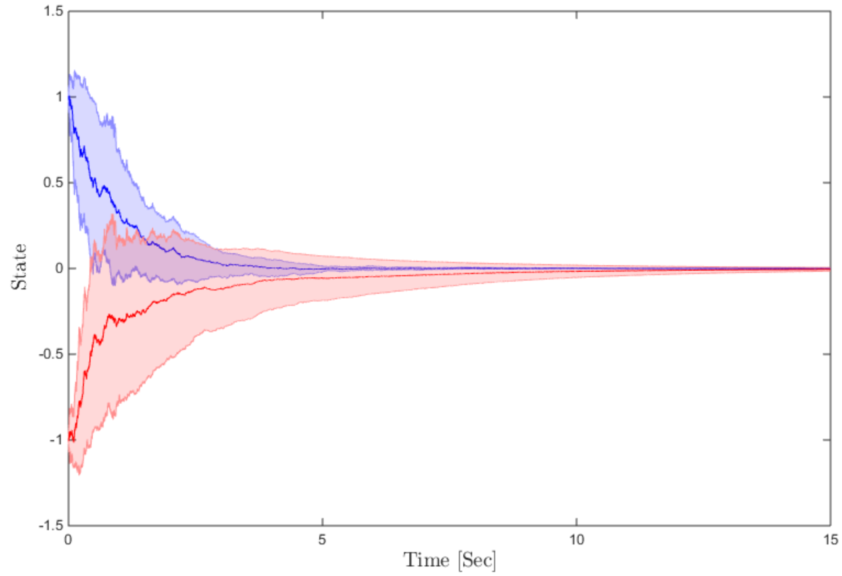

Figure 3.1: Sample average along with the sample standard deviation of the closed-loop system trajectory versus time; $x_1(t)$ in blue and $x_2(t)$ in red.

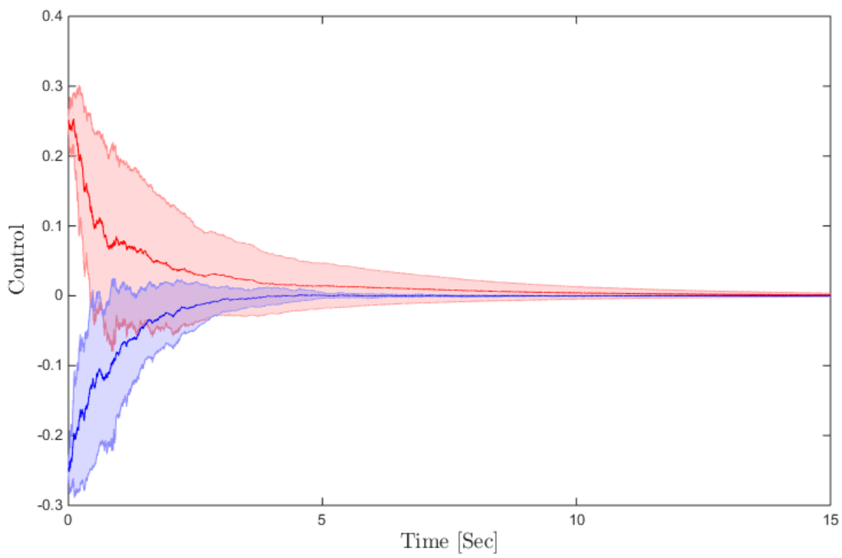

Figure 3.2: Sample average along with the sample standard deviation of the control signal versus time; $u_1(t)$ in blue and $u_2(t)$ in red.

where $x \triangleq [x_1\ x_2\ x_3]^{\mathrm{T}}$, $u \triangleq [u_1\ u_2]^{\mathrm{T}}$, x_1 is the pitch angle, x_2 is the pitch rate, x_3 is the angle of attack, u_1 is the elevator deflection, u_2 is the flaperon deflection, and

$$A = \begin{bmatrix} 0 & 1.00 & 0 \\ 0 & -0.87 & 43.22 \\ 0 & 0.99 & -1.34 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ -17.25 & -1.58 \\ -0.17 & -0.25 \end{bmatrix}, \qquad \sigma = 0.5.$$

In order to design an inverse optimal control law for the controlled stochastic dynamical system (3.97) consider the Lyapunov function candidate given by

$$V(x) = x^{\mathrm{T}}Px + \sum_{q=2}^{3} \frac{1}{q}(x^{\mathrm{T}}M_q x)^q, \tag{3.98}$$

where $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2, 3$. Now, letting $L(x, u) = L_1(x) + u^{\mathrm{T}}R_2 u$, where $R_2 > 0$, it follows from Corollary 3.3 that

$$P = \begin{bmatrix} 0.3773 & 0.0039 & -0.0307 \\ 0.0039 & 0.0032 & 0.0010 \\ -0.0307 & 0.0010 & 0.0906 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0.0740 & -0.0778 & -0.0266 \\ -0.0778 & 0.0836 & 0.0236 \\ -0.0266 & 0.0236 & 0.0354 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0.0005 & -0.0003 & -0.0013 \\ -0.0003 & 0.0008 & -0.0011 \\ -0.0013 & -0.0011 & 0.0140 \end{bmatrix},$$

satisfy (3.83) and (3.84) for $R_1 = 0.3I_3$, $R_2 = 0.01I_2$, $\hat{R}_2 = 0.1I_3$, and

$$\hat{R}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}.$$

In this case, the feedback control law

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}\left(P + \sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right)x$$

globally stabilizes in probability the controlled dynamical system (3.97). Furthermore, the performance measure (3.68), with

$$\begin{aligned} L_1(x) \;=\; & x^{\mathrm{T}}\Bigg(R_1 + \sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}\hat{R}_q + \left[\sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]^{\mathrm{T}} S \\ & \cdot\left[\sum_{q=2}^{3}(x^{\mathrm{T}}M_q x)^{q-1}M_q\right]\Bigg)x, \end{aligned}$$

is minimized in the sense of (3.76).

Figure 6.6 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figure 6.7 shows the sample average along with the standard deviation of the corresponding control signal versus time for $x(0) \overset{\text{a.s.}}{=} [0.5,\ -0.1,\ 0.1]^{\mathrm{T}}$ for 20 sample paths. This controller is compared with the Speyer controller [100] involving a sixth-order cost functional and a cubic-order controller ($q = 2$ in (3.98)) in Figures 3.5 and 3.6. $\triangle$

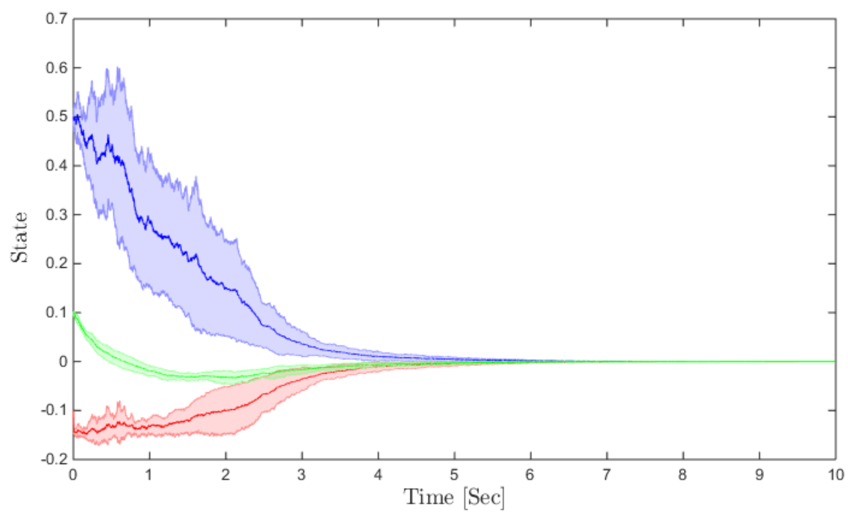

Figure 3.3: Sample average along with the sample standard deviation of the closed-loop system trajectory versus time; $x_1(t)$ in blue, $x_2(t)$ in red, and $x_3(t)$ in green.

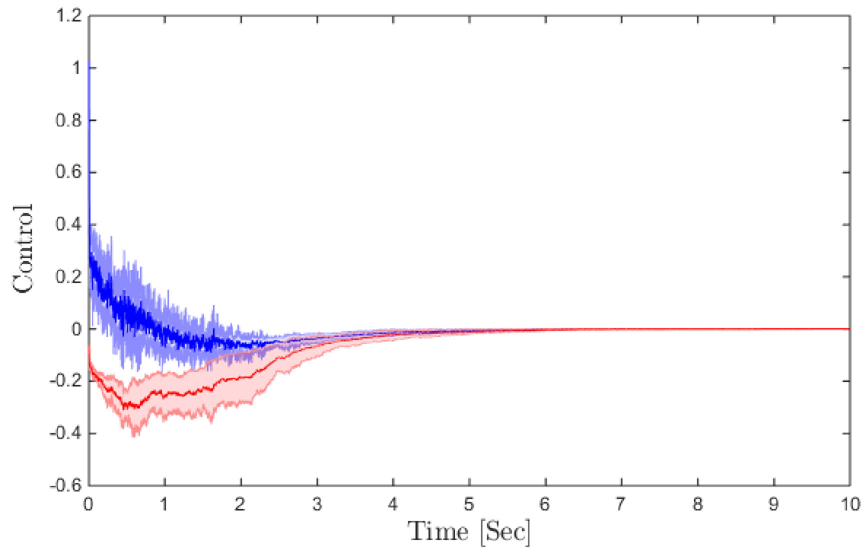

Figure 3.4: Sample average along with the sample standard deviation of the control signal versus time; $u_1(t)$ in blue and $u_2(t)$ in red.

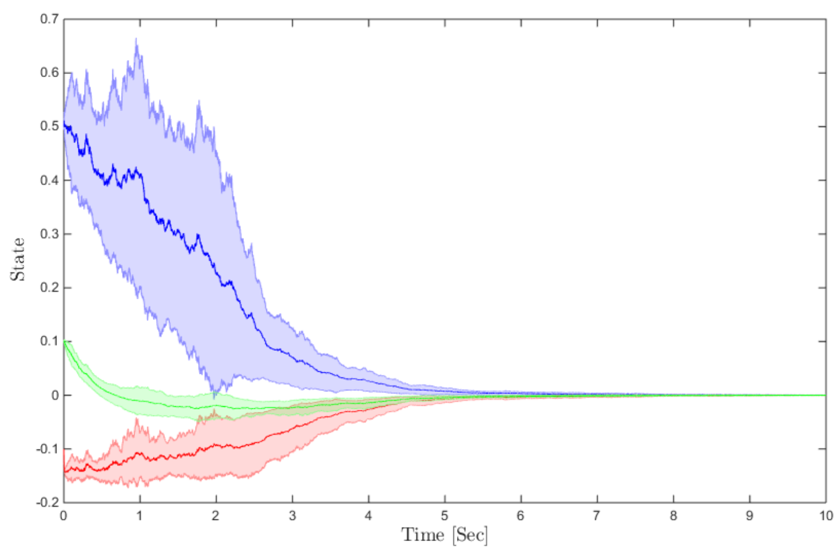

Figure 3.5: Sample average along with the sample standard deviation of the closed-loop system trajectory versus time; $x_1(t)$ in blue, $x_2(t)$ in red, and $x_3(t)$ in green.

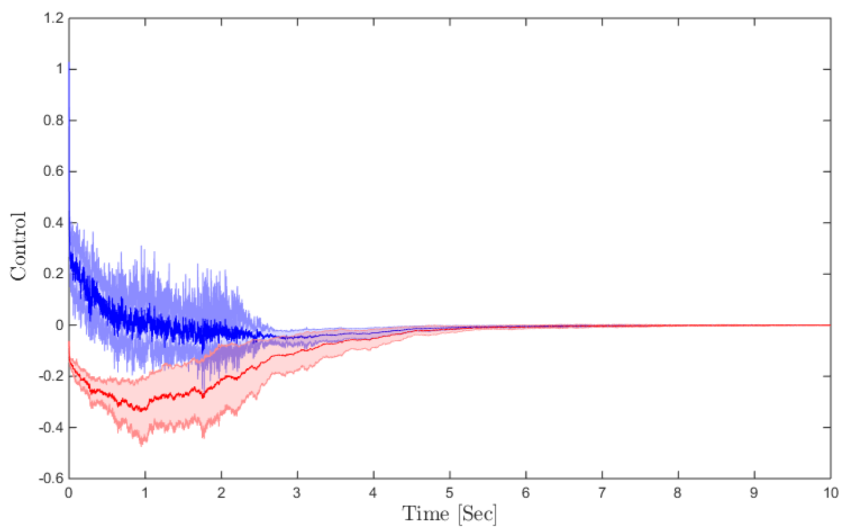

Figure 3.6: Sample average along with the sample standard deviation of the control signal versus time; $u_1(t)$ in blue and $u_2(t)$ in red.

Chapter 4

Partial-State Stabilization and Optimal Feedback Control for Stochastic Dynamical Systems

4.1. Introduction

In this chapter, we extend the framework developed in [69] to address the problem of optimal partial-state *stochastic* stabilization. Specifically, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. In particular, an optimal partial-state stochastic stabilization control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. Another important application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [28, 45]. We exploit this unification and specialize our results to address optimal linear and nonlinear regulation for linear and nonlinear time-varying stochastic systems with quadratic and nonlinear-nonquadratic cost functionals.

More specifically, in Section 4.2, we establish additional notation, definitions, and present some basic results on partial stability of nonlinear stochastic dynamical systems. Then, in Section 4.3, we consider a stochastic nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees partial asymptotic stability in probability. We then state a stochastic optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing partial asymptotic stability in probability of the closed-loop system. These results are then used to address a stochastic optimal control problem for uniform asymptotic stabilization in probability of nonlinear time-varying stochastic dynamical systems.

In Section 4.5, we specialize the results developed in Section 4.3 to affine in the control dynamical systems as well as provide connections to the time-varying, stochastic linear-quadratic regulator problem [64]. In Section 4.5, we develop optimal feedback controllers for affine stochastic nonlinear systems using an inverse optimality framework tailored to the partial-state stochastic stabilization problem. This result is then used to derive time-varying extensions of the results in [11,100] involving nonlinear feedback controllers minimizing polynomial and multilinear performance criteria. In Section 4.6, we provide several illustrative numerical examples that highlight the optimal partial-state stochastic stabilization framework.

4.2. Definitions and Mathematical Preliminaries

In this chapter, we consider nonlinear stochastic autonomous dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x_1(t) = f_1(x_1(t), x_2(t))\mathrm{d}t + D_1(x_1(t), x_2(t))\mathrm{d}w(t), \quad x_1(t_0) \stackrel{\text{a.s.}}{=} x_{10}, \quad t \geq t_0, \tag{4.1}$$

$$\mathrm{d}x_2(t) = f_2(x_1(t), x_2(t))\mathrm{d}t + D_2(x_1(t), x_2(t))\mathrm{d}w(t), \quad x_2(t_0) \stackrel{\text{a.s.}}{=} x_{20}, \tag{4.2}$$

where, for every $t \geq t_0$, $x_1(t) \in \mathcal{H}_{n_1}^{\mathcal{D}}$ and $x_2(t) \in \mathcal{H}_{n_2}$ are such that $x(t) \triangleq \left[x_1^{\mathrm{T}}(t),\ x_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(t_0) \in \mathcal{H}_{n_1}^{\mathcal{D}} \times \mathcal{H}_{n_2}$, $\mathcal{D} \subseteq \mathbb{R}^{n_1}$ is an open set with $0 \in \mathcal{D}$, $w(t)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, and $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is locally Lipschitz continuous in x_1, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz continuous in x_2. In addition, the function $D_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times d}$ is continuous such that, for every $x_2 \in \mathbb{R}^{n_2}$, $D_1(0, x_2) = 0$, and $D_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times d}$ is continuous.

A $\mathbb{R}^{n_1+n_2}$-valued stochastic process $x : [t_0, \tau] \times \Omega \to \mathcal{D} \times \mathbb{R}^{n_2}$ is said to be a *solution* of (4.1) and (4.2) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$ a.s., if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to $\{\mathcal{F}_t\}_{t \geq t_0}$, $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D(x_1, x_2) \triangleq [D_1^{\mathrm{T}}(x_1, x_2), D_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}} \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s)) \mathrm{d}s + \int_{t_0}^{t} D(x(s)) \mathrm{d}w(s) \quad \text{a.s.}, \quad t \in [t_0, \tau], \tag{4.3}$$

where the integrals in (4.3) are Itô integrals. Note that for each fixed $t \geq t_0$, the random variable $\omega \mapsto x(t, \omega)$ assigns a vector $x(\omega)$ to every outcome $\omega \in \Omega$ of an experiment, and for each fixed $\omega \in \Omega$, the mapping $t \mapsto x(t, \omega)$ is the *sample path* of the stochastic process $x(t)$, $t \geq t_0$. A pathwise solution $t \mapsto x(t)$ of (4.1) and (4.2) in $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P}^{x_0})$ is said to be *right maximally* defined if x cannot be extended (either uniquely or non-uniquely) forward in time. We assume that all right maximal pathwise solutions to (4.1) and (4.2) in $(\Omega, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P}^{x_0})$ exist on $[t_0, \infty)$, and hence, we assume that (4.1) and (4.2) is *forward complete*. Sufficient conditions for forward completeness or *global solutions* to (4.1) and (4.2) are given by Corollary 6.3.5 of [6].

Furthermore, we assume that $f : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1+n_2}$ and $D : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{(n_1+n_2) \times d}$ satisfy the uniform Lipschitz continuity condition

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\mathrm{F}} \leq L\|x - y\|, \quad x, y \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{4.4}$$

and the growth restriction condition

$$\|f(x)\|^2 + \|D(x)\|_{\mathrm{F}}^2 \leq L^2(1 + \|x\|^2), \quad x \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{4.5}$$

for some Lipschitz constant $L > 0$, and hence, since $x(t_0) \in \mathcal{H}_{n_1}^{\mathcal{D}} \times \mathcal{H}_{n_2}$ and $x(t_0)$ is independent of $(w(t) - w(t_0))$, $t \geq t_0$, it follows that there exists a unique solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$

of (4.1) and (4.2) in the following sense. For every $x \in \mathcal{H}_{n_1}^{\mathcal{D}} \times \mathcal{H}_{n_2}$ there exists $\tau_x > 0$ such that, if $x_{\mathrm{I}} : [t_0, \tau_1] \times \Omega \to \mathcal{D} \times \mathbb{R}^{n_2}$ and $x_{\mathrm{II}} : [t_0, \tau_2] \times \Omega \to \mathcal{D} \times \mathbb{R}^{n_2}$ are two solutions of (4.1) and (4.2); that is, if $x_{\mathrm{I}}, x_{\mathrm{II}} \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, with continuous sample paths almost surely, solve (4.1) and (4.2), then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $\mathbb{P}\big(x_{\mathrm{I}}(t) = x_{\mathrm{II}}(t),\ t_0 \leq t \leq \tau_x\big) = 1$. Sufficient conditions for forward existence and uniqueness in the absence of the uniform Lipschitz continuity condition and growth restriction condition can be found in [105, 114].

A solution $t \mapsto [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is said to be *regular* if and only if $\mathbb{P}^{x_0}(\tau^{\mathrm{e}} = \infty) = 1$ for all $x(0) \in \mathcal{H}_{n_1}^{\mathcal{D}} \times \mathcal{H}_{n_2}$, where τ^{e} is the first *stopping time* of the solution to (4.1) and (4.2) from every bounded domain in $\mathcal{D} \times \mathbb{R}^{n_2}$. Recall that regularity of solutions imply that solutions exist for $t \geq t_0$ almost surely. Here, we assume regularity of solutions to (4.1) and (4.2), and hence, $\tau_x = \infty$ [67, p.75]. Moreover, the unique solution determines a $\mathbb{R}^{n_1+n_2}$-valued, time-homogeneous Feller continuous Markov process $x(\cdot)$, and hence, its stationary Feller transition probability function is given by $\big($[67, Th. 3.4], [6, Thm. 9.2.8]$\big)$ $\mathbb{P}(x(t) \in B | x(t_0) \stackrel{\text{a.s.}}{=} x_0) = \mathbb{P}(t - t_0, x_0, 0, B)$ for all $x_0 \in \mathcal{D} \times \mathbb{R}^{n_2}$ and $t \geq t_0$, and all Borel subsets $\mathcal{B}$ of $\mathcal{D} \times \mathbb{R}^{n_2}$, where $\mathbb{P}(s, x, t, \mathcal{B}), t \geq s$, denotes the probability of transition of the point $x \in \mathcal{D} \times \mathbb{R}^{n_2}$ at time instant s into the set $\mathcal{B} \subset \mathcal{D} \times \mathbb{R}^{n_2}$ at time instant t. Finally, recall that every continuous process with Feller transition probability function is also a strong Markov process [67, p. 101].

Definition 4.1 [83, Def. 7.7]. Let $x(\cdot)$ be a time-homogeneous Markov process in $\mathcal{H}_{n_1}^{\mathcal{D}} \times \mathcal{H}_{n_2}$ and let $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$. Then the *infinitesimal generator* $\mathcal{L}$ of $x(t)$, $t \geq 0$, with $x(0) = x_0$ a.s., is defined by

$$\mathcal{L}V(x_0) \triangleq \lim_{t \to 0^+} \frac{\mathbb{E}^{x_0}[V(x(t))] - V(x_0)}{t}, \quad x_0 \in \mathcal{D} \times \mathbb{R}^{n_2}. \tag{4.6}$$

If $V \in \mathrm{C}^2$ and has a compact support, and $x(t)$, $t \geq t_0$, satisfies (4.1) and (4.2), then the limit in (4.6) exists for all $x \in \mathcal{D} \times \mathbb{R}^{n_2}$ and the infinitesimal generator $\mathcal{L}$ of $x(t)$, $t \geq t_0$, can be characterized by the system *drift* and *diffusion* functions $f(x)$ and $D(x)$ defining the stochastic dynamical system (4.1) and (4.2) with system state $x(t)$, $t \geq t_0$, and is given by ([83, Thm. 7.9])

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2} \mathrm{tr}\; D^{\mathrm{T}}(x) \frac{\partial^2 V(x)}{\partial x^2} D(x), \quad x \in \mathcal{D} \times \mathbb{R}^{n_2}. \tag{4.7}$$

In the following definition we introduce the notion of stochastic partial stability.

Definition 4.2. *i*) The nonlinear stochastic dynamical system $\mathcal{G}$ given by (4.1) and (4.2) is *Lyapunov stable in probability with respect to* x_1 *uniformly in* x_{20} if, for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\rho, \varepsilon) > 0$ such that, for all $x_{10} \in \mathcal{B}_\delta(0)$,

$$\mathbb{P}^{x_0}\left(\sup_{t \geq t_0} \|x_1(t)\| \leq \varepsilon\right) \geq 1 - \rho \tag{4.8}$$

for all $t \geq 0$ and all $x_{20} \in \mathbb{R}^{n_2}$.

ii) $\mathcal{G}$ is *asymptotically stable in probability with respect to* x_1 *uniformly in* x_{20} if $\mathcal{G}$ is Lyapunov stable in probability with respect to x_1 uniformly in x_{20} and

$$\lim_{x_{10} \to 0} \mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x_1(t)\| = 0\right) = 1 \tag{4.9}$$

uniformly in x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

iii) $\mathcal{G}$ is *globally asymptotically stable in probability with respect to* x_1 *uniformly in* x_{20} if $\mathcal{G}$ is Lyapunov stable in probability with respect to x_1 uniformly in x_{20} and $\mathbb{P}^{x_0}\left(\lim_{t \to \infty} \|x_1(t)\| = 0\right) = 1$ holds uniformly in x_{20} for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Remark 4.1. It is important to note that there is a key difference between the stochastic partial stability definitions given in Definitions 4.2 and the definitions of stochastic partial stability given in [96]. In particular, the stochastic partial stability definitions given in [96] require that both the initial conditions x_{10} and x_{20} lie in a neighborhood of origin, whereas in Definition 4.2 x_{20} can be arbitrary. As will be seen below, this difference allows us to unify autonomous stochastic partial stability theory with time-varying stochastic stability theory.

An additional difference between our formulation of the stochastic partial stability problem and the stochastic partial stability problem considered in [96] is in the treatment of the equilibrium of (4.1) and (4.2). Specifically, in our formulation we require the weaker partial equilibrium condition $f_1(0, x_2) = 0$ and $D_1(0, x_2) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$, whereas in [96] the author requires the stronger equilibrium condition $f_1(0, 0) = 0, f_2(0, 0) = 0, D_1(0, 0) = 0$, and $D_2(0, 0) = 0$.

Remark 4.2. As far as the analysis and synthesis problem considered in Chapter 3, a more general stochastic stability notion can also be introduced here involving stochastic stability and convergence to an invariant (stationary) distribution. In this case, state convergence is not to an equilibrium point but rather to a stationary distribution. This framework can relax the vanishing perturbation assumption $D_1(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and requires a more involved analysis and synthesis framework showing stability of the underlying Markov semigroup [78].

As shown in [45] and [28], an important application of deterministic partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. A similar unification can be provided for stochastic dynamical systems. Specifically, consider the nonlinear time-varying stochastic dynamical system given by

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + D(t, x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq t_0, \tag{4.10}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f(t, 0) = 0$, $D(t, 0) = 0$, and $f : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ and $D : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}^{n \times d}$ are jointly continuous in t and x, and satisfy (4.4) and (4.5) for all $x \in \mathcal{D}$ uniformly in t for all t in compact subsets of $[t_0, \infty)$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$ a.s., where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \geq t_0$, to the nonlinear time-varying stochastic dynamical system (4.10) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \geq 0$, to the nonlinear autonomous stochastic dynamical system

$$\mathrm{d}x_1(\tau) = f(x_2(\tau), x_1(\tau))\mathrm{d}\tau + D(x_2(\tau), x_1(\tau))\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_0, \quad \tau \geq 0, \tag{4.11}$$

$$\mathrm{d}x_2(\tau) = \mathrm{d}\tau, \qquad x_2(0) \overset{\text{a.s.}}{=} t_0. \tag{4.12}$$

Note that (4.11) and (4.12) are in the same form as the system given by (4.1) and (4.2), and Definition 4.2 applied to (4.11) and (4.12) specializes to the definitions of uniform Lyapunov stability in probability, uniform asymptotic stability in probability, and global uniform asymptotic stability in probability of (4.10); for details see [6, 68].

Next, we provide sufficient conditions for partial stability of the nonlinear stochastic dynamical system given by (4.1) and (4.2).

Theorem 4.1. Consider the nonlinear stochastic dynamical system (4.1) and (4.2). Then the following statements hold:

i) If there exist a two-times continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that, for all $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$,

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \tag{4.13}$$

$$\frac{\partial V(x_1, x_2)}{\partial x_1} f_1(x_1, x_2) + \frac{\partial V(x_1, x_2)}{\partial x_2} f_2(x_1, x_2) + \frac{1}{2}\mathrm{tr}\ D_1^{\mathrm{T}}(x_1, x_2)\frac{\partial^2 V(x_1, x_2)}{\partial x_1^2} D_1(x_1, x_2)$$
$$+\frac{1}{2}\mathrm{tr}\ D_2^{\mathrm{T}}(x_1, x_2)\frac{\partial^2 V(x_1, x_2)}{\partial x_2^2} D_2(x_1, x_2) \leq -\gamma(\|x_1\|), \tag{4.14}$$

then the nonlinear dynamical system given by (4.1) and (4.2) is asymptotically stable in probability with respect to x_1 uniformly in x_{20}.

ii) If there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ satisfying (4.13) and (4.14), then the nonlinear dynamical system given by (4.1) and (4.2) is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20}.

Proof: *i*) Let $x_{20} \in \mathbb{R}^{n_2}$, let $\varepsilon > 0$ be such that $\mathcal{B}_\varepsilon(0) \subseteq \mathcal{D}$, let $\rho > 0$, and define $\mathcal{D}_{\varepsilon,\rho} \triangleq \{x_1 \in \mathcal{B}_\varepsilon(0) : V(x_1, x_{20}) < \alpha(\varepsilon)\rho\}$. Since $V(\cdot,\cdot)$ is continuous and $V(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, it follows that $\mathcal{D}_{\varepsilon,\rho}$ is nonempty and there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that $V(x_1, x_{20}) < \alpha(\varepsilon)\rho$, $x_1 \in \mathcal{B}_\delta(0)$. Hence, $\mathcal{B}_\delta(0) \subseteq \mathcal{D}_{\varepsilon,\rho}$. Next, it follows from (4.14) that $V(x_1(t), x_2(t))$ is a (positive) supermartingale [67, Lemma 5.4], and hence, for every $x_1(0) \in \mathcal{H}_{n_1}^{\mathcal{B}_\delta(0)} \subseteq \mathcal{H}_{n_1}^{\mathcal{D}_{\varepsilon,\rho}}$, it follows from (4.13) and the extended version of the Markov inequality for monotonically increasing functions [41, p. 193] that

$$\begin{aligned}\mathbb{P}^{x_0}\left(\sup_{t\geq 0}\|x_1(t)\| \geq \varepsilon\right) &\leq \sup_{t\geq 0}\frac{\mathbb{E}^{x_0}[\alpha(\|x_1(t)\|)]}{\alpha(\varepsilon)} \\ &\leq \sup_{t\geq 0}\frac{\mathbb{E}^{x_0}[V(x_1(t), x_2(t))]}{\alpha(\varepsilon)} \\ &\leq \frac{\mathbb{E}^{x_0}[V(x_1(0), x_2(0))]}{\alpha(\varepsilon)} \\ &\leq \rho,\end{aligned}$$

which proves partial Lyapunov stability in probability with respect to x_1 uniformly in x_{20}.

To prove partial asymptotic stability in probability with respect to x_1, note that it follows from (4.13) and (4.14) that

$$\mathcal{L}V(x_1, x_2) \leq -\gamma(\|x_1\|) \leq -\gamma \circ \beta^{-1}(V(x_1, x_2)), \quad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}.$$

Furthermore, it follows from partial Lyapunov stability in probability that $\mathcal{B}_\varepsilon(0) \times \mathbb{R}^{n_2}$ is an invariant set with respect to the solutions of (4.1) and (4.2) as $\varepsilon \to 0$, and hence, using Corollary 4.2 of [75] with $\eta(\cdot) = \gamma \circ \beta^{-1}(\cdot)$ it follows that $\lim_{t\to\infty} \gamma \circ \beta^{-1}(V(x_1(t), x_2(t))) \overset{\text{a.s.}}{=} 0$. Furthermore, using the properties of the class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ it follows that $\lim_{t\to\infty} V\ (x_1(t), x_2(t)) \overset{\text{a.s.}}{=} 0$, which yields $\lim_{t\to\infty} \alpha(\|x_1(t)\|) \leq \lim_{t\to\infty} V(x_1(t),\ x_2(t)) \overset{\text{a.s.}}{=} 0$. Hence, $\lim_{t\to\infty} x_1(t) \overset{\text{a.s.}}{\to} 0$ as $x_{10} \to 0$, which proves partial asymptotic stability in probability with respect to x_1 uniformly in x_{20}.

ii) Finally, for $\mathcal{D} = \mathbb{R}^{n_1}$ globally asymptotically stable in probability with respect to x_1 uniformly in x_{20} is direct consequence of the radially unbounded condition on $V(\cdot, \cdot)$ using standard arguments and the fact that $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions. □

4.3. Stochastic Optimal Partial-State Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the stochastic nonlinear dynamical system given by (4.1) and (4.2). In particular, we show that the nonlinear-nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t\right], \tag{4.15}$$

where $L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (4.1) and (4.2), can be evaluated in a convenient form so long as (4.1) and (4.2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and proves asymptotic stability in probability of (4.1) and (4.2) with respect to x_1 uniformly in x_{20}.

Theorem 4.2. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (4.1) and (4.2) with performance measure (4.15). Assume that there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \tag{4.16}$$

$$\frac{\partial V(x_1,x_2)}{\partial x_1} f_1(x_1,x_2) + \frac{\partial V(x_1,x_2)}{\partial x_2} f_2(x_1,x_2) + \frac{1}{2}\mathrm{tr}\ D_1^{\mathrm{T}}(x_1,x_2)\frac{\partial^2 V(x_1,x_2)}{\partial x_1^2} D_1(x_1,x_2) + \frac{1}{2}\mathrm{tr}\ D_2^{\mathrm{T}}(x_1,x_2)\frac{\partial^2 V(x_1,x_2)}{\partial x_2^2} D_2(x_1,x_2) \leq -\gamma(\|x_1\|), \tag{4.17}$$

$$L(x_1,x_2) + \frac{\partial V(x_1,x_2)}{\partial x_1} f_1(x_1,x_2) + \frac{\partial V(x_1,x_2)}{\partial x_2} f_2(x_1,x_2) + \frac{1}{2}\mathrm{tr}\ D_1^{\mathrm{T}}(x_1,x_2)\frac{\partial^2 V(x_1,x_2)}{\partial x_1^2} D_1(x_1,x_2) + \frac{1}{2}\mathrm{tr}\ D_2^{\mathrm{T}}(x_1,x_2)\frac{\partial^2 V(x_1,x_2)}{\partial x_2^2} D_2(x_1,x_2) = 0. \tag{4.18}$$

Then the nonlinear stochastic dynamical system $\mathcal{G}$ is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20} and, for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \tag{4.19}$$

Proof: Let $x_1(t)$ and $x_2(t)$, $t \geq t_0$, satisfy (4.1) and (4.2). Then (4.16) and (4.17) are a restatement (4.13) and (4.14), and hence, it follows from Theorem 4.1 that the system $\mathcal{G}$ is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20}. Consequently, $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \|x_1(t)\| = 0\right) = 1$ holds for all initial conditions $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Next, using Itô's (chain rule) formula, it follows that the stochastic differential of $V(x_1(t), x_2(t))$ along the system trajectories $x_1(t)$ and $x_2(t)$, $t \geq t_0$, is given by

$$\begin{aligned}
\mathrm{d}V(x_1(t), x_2(t)) = &\bigg(\frac{\partial V(x_1(t), x_2(t))}{\partial x_1} f_1(x_1(t), x_2(t)) + \frac{\partial V(x_1(t), x_2(t))}{\partial x_2} f_2(x_1(t), x_2(t)) \\
&+ \frac{1}{2}\mathrm{tr}\ D_1^{\mathrm{T}}(x_1(t), x_2(t)) \frac{\partial^2 V(x_1(t), x_2(t))}{\partial x_1^2} D_1(x_1(t), x_2(t)) \\
&+ \frac{1}{2}\mathrm{tr}\ D_2^{\mathrm{T}}(x_1(t), x_2(t)) \frac{\partial^2 V(x_1(t), x_2(t))}{\partial x_2^2} D_2(x_1(t), x_2(t))\bigg)\mathrm{d}t \\
&+ \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t).
\end{aligned} \tag{4.20}$$

Hence, using (4.18) it follows that

$$\begin{aligned}
&L(x_1(t), x_2(t))\mathrm{d}t + \mathrm{d}V(x_1(t), x_2(t)) \\
&\quad = \bigg(L(x_1(t), x_2(t)) + \frac{\partial V(x_1(t), x_2(t))}{\partial x_1} f_1(x_1(t), x_2(t)) + \frac{\partial V(x_1(t), x_2(t))}{\partial x_2} f_2(x_1(t), x_2(t)) \\
&\qquad + \frac{1}{2}\mathrm{tr}\ D_1^{\mathrm{T}}(x_1(t), x_2(t)) \frac{\partial^2 V(x_1(t), x_2(t))}{\partial x_1^2} D_1(x_1(t), x_2(t)) \\
&\qquad + \frac{1}{2}\mathrm{tr}\ D_2^{\mathrm{T}}(x_1(t), x_2(t)) \frac{\partial^2 V(x_1(t), x_2(t))}{\partial x_2^2} D_2(x_1(t), x_2(t))\bigg)\mathrm{d}t \\
&\qquad + \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t) \\
&\quad = \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t).
\end{aligned} \tag{4.21}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [t_0, \infty)$ be the first exit (stopping) time of the solution $x_1(t)$ and $x_2(t)$, $t \geq t_0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty}\tau_m$. Now, integrating (4.21) over $[t_0, \min\{t_n, \tau_m\}]$, where $(n,m)\in\mathbb{Z}_+\times\mathbb{Z}_+$, yields

$$
\begin{aligned}
\int_{t_0}^{\min\{t_n,\tau_m\}} & L(x_1(t),x_2(t))\ \mathrm{d}t \\
&= -\int_{t_0}^{\min\{t_n,\tau_m\}} \mathrm{d}V(x_1(t),x_2(t)) + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t),x_2(t))\mathrm{d}w(t) \\
&= V(x_1(t_0),x_2(t_0)) - V(x_1(\min\{t_n,\tau_m\}),x_2(\min\{t_n,\tau_m\})) \\
&\quad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t),x_2(t))\mathrm{d}w(t). \qquad (4.22)
\end{aligned}
$$

Next, taking the expectation on both sides of (4.22) yields

$$
\begin{aligned}
\mathbb{E}^{x_0}&\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t))\ \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\Bigg[V(x_1(t_0),x_2(t_0)) - V(x_1(\min\{t_n,\tau_m\}),x_2(\min\{t_n,\tau_m\})) \\
&\quad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t),x_2(t))\mathrm{d}w(t)\Bigg] \\
&= V(x_{10},x_{20}) - \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n,\tau_m\}),x_2(\min\{t_n,\tau_m\}))\right]. \qquad (4.23)
\end{aligned}
$$

Now, noting that $L(x_1,x_2)\geq 0$, $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, the sequence of random variables $\{f_{n,m}\}_{n,m=0}^{\infty}\subseteq\mathcal{H}_1$, where

$$
f_{n,m} \triangleq \int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t))\mathrm{d}t,
$$

is a pointwise nondecreasing sequence in n and m of nonnegative $\mathcal{F}_t$-measurable random variables on Ω. Moreover, defining the improper integral

$$
\int_{t_0}^{\infty} L(x_1(t),x_2(t))\mathrm{d}t
$$

as the limit of a sequence of proper integrals, it follows from the Lebesgue monotone convergence theorem [3] that

$$
\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{E}^{x_0}&\left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t))\ \mathrm{d}t\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t))\ \mathrm{d}t\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_{t_0}^{\tau_m} L(x_1(t), x_2(t))\ \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x_1(t), x_2(t))\ \mathrm{d}t\right] \\
&= J(x_{10}, x_{20}).
\end{aligned} \tag{4.24}
$$

Next, since $\mathcal{G}$ is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20}, $V(\cdot,\cdot)$ is continuous, $V(x_1(t), x_2(t))$, $t \geq t_0$, is positive supermartingale by (4.17) and [67, Lemma 5.4] it follows from [67, Theorem 5.1] that

$$
\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right] \\
&\qquad = \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right] \\
&\qquad = \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right].
\end{aligned} \tag{4.25}
$$

Now, it follows from (4.16) that

$$
\begin{aligned}
&V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty}\beta\left(\|x_1(\min\{t_n,\tau_m\})\|\right)\right] \\
&\qquad \leq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right] \\
&\qquad \leq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty}\alpha\left(\|x_1(\min\{t_n,\tau_m\})\|\right)\right],
\end{aligned} \tag{4.26}
$$

and hence, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (4.23), using (4.24) and (4.25), and using the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$, we obtain

$$
\begin{aligned}
&V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\beta\left(\lim_{m\to\infty}\lim_{n\to\infty}\|x_1(\min\{t_n,\tau_m\})\|\right)\right] \\
&\qquad \leq J(x_{10}, x_{20}) \leq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\alpha\left(\lim_{m\to\infty}\lim_{n\to\infty}\|x_1(\min\{t_n,\tau_m\})\|\right)\right].
\end{aligned} \tag{4.27}
$$

Finally, using $\mathbb{P}^{x_0}\left(\lim_{t\to\infty}\|x_1(t)\| = 0\right) = 1$ for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, (4.19) is a direct consequence of (4.27). □

The following corollary to Theorem 4.2 considers the nonautonomous stochastic dynamical system (4.10) with performance measure

$$
J(t_0, x_0) \triangleq \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(t, x(t))\mathrm{d}t\right], \tag{4.28}
$$

where $L : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous in t and x, and $x(t)$, $t \geq t_0$, satisfies (4.10).

Corollary 4.1. Consider the nonlinear time-varying stochastic dynamical system (4.10) with performance measure (4.28). Assume that there exist a two-times continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$,

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \tag{4.29}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x) \frac{\partial^2 V(t,x)}{\partial x^2} D(t,x) \leq -\gamma(\|x\|), \tag{4.30}$$

$$-\frac{\partial V(t,x)}{\partial t} = L(t,x) + \frac{\partial V(t,x)}{\partial x} f(t,x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x) \frac{\partial^2 V(t,x)}{\partial x^2} D(t,x). \tag{4.31}$$

Then the stochastic nonlinear dynamical system (4.10) is globally uniformly asymptotically stable in probability and $J(t_0, x_0) = V(t_0, x_0)$ for all $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^n$.

Proof: The result is a direct consequence of Theorem 4.2 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, $D_1(x_1, x_2) = D_1(x_2, x_1) = D(t, x)$, $D_2(x_1, x_2) = 0$, and $V(x_1, x_2) = V(x_2, x_1) = V(t, x)$. □

Next, we use the framework developed in Theorem 4.2 to obtain a characterization of stochastic optimal feedback controllers that guarantee closed-loop, partial-state stabilization in probability. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the stochastic Hamilton-Jacobi-Bellman equation. To address the problem of characterizing partially stabilizing feedback controllers, consider the nonlinear controlled stochastic dynamical system

$$\mathrm{d}x_1(t) = F_1(x_1(t), x_2(t), u(t))\mathrm{d}t + D_1(x_1(t), x_2(t), u(t))\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{4.32}$$

$$\mathrm{d}x_2(t) = F_2(x_1(t), x_2(t), u(t))\mathrm{d}t + D_2(x_1(t), x_2(t), u(t))\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{4.33}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{H}_{n_1}$, $x_2(t) \in \mathcal{H}_{n_2}$, $u(t) \in \mathcal{H}_m$, $F_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1}$, $F_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2}$, $D_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1 \times d}$, $D_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2 \times d}$, and $F_1(0, x_2, 0) = 0$ and $D_1(0, x_2, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$.

Here we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that (4.32) and (4.33) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (4.32) and (4.33) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq 0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau), w(\tau), \tau \leq s$, and $x(0) = [x_1^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0)]^{\mathrm{T}}$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity and growth conditions (4.4) and (4.5) hold for the controlled drift and diffusion terms $F(x_1, x_2, u) \triangleq [F_1^{\mathrm{T}}(x_1, x_2, u), F_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$ and $D(x_1, x_2, u) \triangleq [D_1^{\mathrm{T}}(x_1, x_2, u), D_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$ uniformly in u. In this case, it follows from Theorem 2.2.4 of [5] that there exists a pathwise unique solution to (4.32) and (4.33) in $(\Omega, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, is called a *control law*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, where $\phi(\cdot, \cdot)$ is a control law and $x_1(t)$ and $x_2(t)$ satisfy (4.32) and (4.33), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(x_1(t), x_2(t)) \in \mathcal{H}_m$, $t \geq 0$. Given a control law $\phi(\cdot, \cdot)$ and a feedback control law $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, the *closed-loop system* (4.32) and (4.33) is given by

$$\mathrm{d}x_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t)))\mathrm{d}t + D_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t)))\mathrm{d}w(t),$$
$$x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \qquad (4.34)$$
$$\mathrm{d}x_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t)))\mathrm{d}t + D_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t)))\mathrm{d}w(t),$$
$$x_2(0) \overset{\text{a.s.}}{=} x_{20}. \qquad (4.35)$$

Next, we present a main theorem for partial-state stabilization in probability characterizing feedback controllers that guarantee partial closed-loop stability in probability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, let $L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$ be jointly continuous in x_1, x_2, and u, and define the set of partial regulation controllers given by

$$\mathcal{S}(x_1(0), x_2(0)) \triangleq \Big\{u(\cdot) : u(\cdot) \text{ is admissible and } x_1(\cdot) \text{ given by (4.32)} \\ \text{satisfies } \mathbb{P}^{x_0}\Big(\lim_{t\to\infty} \|x_1(t)\| = 0\Big) = 1\Big\}.$$

Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, that is, inputs corresponding to partial-state null convergent in probability solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost.

Theorem 4.3. Consider the nonlinear controlled stochastic dynamical system $\mathcal{G}$ given by (4.32) and (4.33) with performance functional

$$J(x_{10}, x_{20}, u(\cdot)) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty L(x_1(t), x_2(t), u(t))\,\mathrm{d}t\right], \tag{4.36}$$

where $u(\cdot)$ is an admissible control. Assume that there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a class $\mathcal{K}$ function $\gamma(\cdot)$, and a control law $\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \beta(\|x_1\|), \tag{4.37}$$

$$V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x_1, x_2, \phi(x_1, x_2))V''(x_1, x_2) \\ \cdot D(x_1, x_2, \phi(x_1, x_2)) \le -\gamma(\|x_1\|), \tag{4.38}$$

$$\phi(0, x_2) = 0, \tag{4.39}$$

$$H(x_1, x_2, \phi(x)) = 0, \tag{4.40}$$

$$H(x_1, x_2, u) \ge 0, \qquad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \tag{4.41}$$

where

$$H(x_1, x_2, u) \triangleq L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u) \\ + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x_1, x_2, u)V''(x_1, x_2)D(x_1, x_2, u). \tag{4.42}$$

Then, with the feedback control $u = \phi(x_1, x_2)$, the closed-loop system given by (4.34) and (4.35) is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20} and

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{4.43}$$

In addition, if $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then the feedback control $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ minimizes $J(x_{10}, x_{20}, u(\cdot))$ in the sense that

$$J(x_{10}, x_{20}, \phi(\cdot, \cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))} J(x_{10}, x_{20}, u(\cdot)). \tag{4.44}$$

Proof: Global asymptotic stability in probability with respect to x_1 uniformly in x_{20} is a direct consequence of (4.37) and (4.38) by applying Theorem 4.1 to the closed-loop system given by (4.34) and (4.35). Furthermore, using (4.40), condition (4.43) is a restatement of (4.19) as applied to the closed-loop system.

Next, let $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, let $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, and let $x_1(t)$ and $x_2(t)$, $t \geq 0$, be solutions of (4.32) and (4.33). Then, using Itô's (chain rule) formula, the stochastic differential of $V(x_1(t), x_2(t))$ along the system trajectories $(x_1(t), x_2(t)), t \geq 0$, is given by

$$\mathrm{d}V(x_1(t), x_2(t)) = \mathcal{L}V(x_1(t), x_2(t))\mathrm{d}t + \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t), u(t))\mathrm{d}w(t). \tag{4.45}$$

Hence, using (4.7) and (4.42) yields

$$\begin{aligned} L(x_1(t), x_2(t), u(t))\mathrm{d}t = & -\mathrm{d}V(x_1(t), x_2(t)) + (L(x_1(t), x_2(t), u(t)) + \mathcal{L}V(x_1(t), x_2(t)))\,\mathrm{d}t \\ & + \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t), u(t))\mathrm{d}w(t) \\ = & -\mathrm{d}V(x_1(t), x_2(t)) + H(x_1(t), x_2(t), u(t))\mathrm{d}t \\ & + \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t), u(t))\mathrm{d}w(t). \end{aligned} \tag{4.46}$$

Now, it follows from (4.37) that

$$\mathbb{E}^{x_0}\left[\lim_{t\to\infty} \alpha(\|x_1(t)\|)\right] \leq \mathbb{E}^{x_0}\left[\lim_{t\to\infty} V(x_1(t), x_2(t))\right] \leq \mathbb{E}^{x_0}\left[\lim_{t\to\infty} \beta(\|x_1(t)\|)\right]. \tag{4.47}$$

Using the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$, and the fact that $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \|x_1(t)\| = 0\right) = 1$ for all $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, it follows from (4.47) that

$$0 = \mathbb{E}^{x_0}\left[\alpha\left(\lim_{t\to\infty} \|x_1(t)\|\right)\right] \leq \mathbb{E}^{x_0}\left[\lim_{t\to\infty} V(x_1(t), x_2(t))\right] \leq \mathbb{E}^{x_0}\left[\beta\left(\lim_{t\to\infty} \|x_1(t)\|\right)\right] = 0. \tag{4.48}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [0, \infty)$ be the first exit (stopping) time of the solution $x_1(t)$ and $x_2(t)$, $t \geq 0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, integrating (4.46) over $[t_0, \min\{t_n, \tau_m\}]$, where $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\begin{aligned}
\int_0^{\min\{t_n,\tau_m\}} & L(x_1(t), x_2(t), u(t))\ \mathrm{d}t \\
&= -\int_0^{\min\{t_n,\tau_m\}} \mathrm{d}V(x_1(t), x_2(t)) + \int_0^{\min\{t_n,\tau_m\}} H(x_1(t), x_2(t), u(t))\mathrm{d}t \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t) \\
&= V(x_1(0), x_2(0)) - V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\})) \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} H(x_1(t), x_2(t), u(t))\mathrm{d}t \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t).
\end{aligned} \tag{4.49}$$

Next, taking the expectation on both sides of (4.49) and using (4.41) yields

$$\begin{aligned}
\mathbb{E}^{x_0} & \left[\int_0^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t), u(t))\ \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\Bigg[V(x_1(0), x_2(0)) - V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\})) \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} H(x_1(t), x_2(t), u(t))\mathrm{d}t \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t)\Bigg] \\
&= V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\}))\right] \\
&\quad + \mathbb{E}^{x_0}\left[\int_0^{\min\{t_n,\tau_m\}} H(x_1(t), x_2(t), u(t))\mathrm{d}t\right] \\
&\geq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\}))\right].
\end{aligned} \tag{4.50}$$

Now, noting that for all $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$,

$$\int_0^{\infty} |L(x_1(t), x_2(t), u(t))|\ \mathrm{d}t \overset{\text{a.s.}}{<} \infty,$$

define the random variable

$$g \triangleq \sup_{t\geq 0, m>0} \int_0^{\min\{t,\tau_m\}} |L(x_1(s), x_2(s), u(s))|\ \mathrm{d}s.$$

In this case, the sequence of $\mathcal{F}_t$-measurable random variables $\{f_{n,m}\}_{n,m=0}^{\infty} \subseteq \mathcal{H}_1$ on Ω, where

$$f_{n,m} \triangleq \int_0^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t), u(t))\mathrm{d}t,$$

satisfies $|f_{n,m}| \overset{\text{a.s.}}{<} g$.

Next, defining the improper integral

$$\int_0^{\infty} L(x_1(t), x_2(t), u(t))\mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from dominated convergence theorem [3] that

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0}&\left[\int_0^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t), u(t))\mathrm{d}t\right]\\ &= \lim_{m\to\infty} \mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_0^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t), u(t))\mathrm{d}t\right]\\ &= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_0^{\tau_m} L(x_1(t), x_2(t), u(t))\mathrm{d}t\right]\\ &= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x_1(t), x_2(t), u(t))\mathrm{d}t\right]\\ &= J(x_{10}, x_{20}, u(\cdot)).\end{aligned} \tag{4.51}$$

Finally, using the fact that $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$ and $V(\cdot,\cdot)$ is continuous, it follows that for every $m > 0$, $V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))$ is bounded for all $\{t_n\}_{n=0}^{\infty}$. Thus, using the dominated convergence theorem we obtain

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0}&\left[V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right]\\ &= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right].\end{aligned} \tag{4.52}$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (4.50) and using the fact $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, (4.48), (4.51), (4.52), and $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20})$ yields (4.44). □

Note that (4.40) is the steady-state, stochastic Hamilton-Jacobi-Bellman equation for the nonlinear controlled stochastic dynamical system (4.32) and (4.33) with performance criterion (4.36). Furthermore, conditions (4.40) and (4.41) guarantee optimality with respect to the set of admissible partially asymptotically stabilizing in probability controllers $\mathcal{S}(x_0(0), x_2(0))$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_1(0), x_2(0))$ is not required. In addition, the stochastic optimal asymptotically stabilizing in probability with respect to x_1 uniformly in x_{20} *feedback* control law $u = \phi(x_1, x_2)$ is independent of the initial condition (x_{10}, x_{20}) and is given by

$$\phi(x_1, x_2) = \underset{u\in\mathcal{S}(x_1(0),x_2(0))}{\arg\min} \Big[L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u) + \frac{1}{2}\text{tr}\; D^{\mathrm{T}}(x_1, x_2, u)V''(x_1, x_2)D(x_1, x_2, u) \Big]. \tag{4.53}$$

Remark 4.3. Setting $n_1 = n$ and $n_2 = 0$, the nonlinear controlled stochastic dynamical system given by (4.32) and (4.33) reduces to

$$\mathrm{d}x(t) = F(x(t), u(t))\mathrm{d}t + D(x(t), u(t))\mathrm{d}w(t), \qquad x(0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq 0. \tag{4.54}$$

In this case, (4.37) implies that $V(\cdot)$ is positive definite with respect to x and the conditions of Theorem 4.3 reduce to the conditions given in Chapter 4 of [68] characterizing the classical stochastic optimal control problem for time-invariant systems on an infinite interval.

Finally, we use Theorem 4.3 to provide a unification between optimal partial-state stochastic stabilization and stochastic optimal control for nonlinear time-varying systems. Specifically, consider the nonlinear time-varying controlled stochastic dynamical system

$$\mathrm{d}x(t) = F(t, x(t), u(t))\mathrm{d}t + D(t, x(t), u(t))\mathrm{d}w(t), \qquad x(t_0) = x_0 \quad \text{a.s.}, \qquad t \geq t_0, \tag{4.55}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) \triangleq \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(t, x(t), u(t))\mathrm{d}t\right], \tag{4.56}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n$, $u(t) \in \mathcal{H}_m$, $L : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $F : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $D : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times d}$ are jointly continuous in t, x, and u, $F(t, \cdot, u)$

and $D(t,\cdot,u)$ is Lipschitz continuous in x for every $(t,u)\in[t_0,\infty)\times\mathbb{R}^m$, and $F(t,x,\cdot)$ and $D(t,x,\cdot)$ is Lipschitz continuous in u for every $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0,x(t_0))\triangleq\Big\{u(\cdot):u(\cdot)\text{ is admissible and }x(\cdot)\text{ given by (4.55)}\\ \text{satisfies }\mathbb{P}^{x_0}\Big(\lim_{t\to\infty}\|x(t)\|=0\Big)=1\Big\}.$$

Corollary 4.2. Consider the nonlinear time-varying controlled stochastic dynamical system (4.55) with performance measure (4.56) where $u(\cdot)$ is an admissible control. Assume that there exist a two-times continuously differentiable function $V:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a class $\mathcal{K}$ function $\gamma(\cdot)$, and a control law $\phi:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^m$ such that, for all $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$,

$$\alpha(\|x\|)\le V(t,x)\le\beta(\|x\|), \tag{4.57}$$

$$\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}F(t,x,\phi(t,x))+\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,\phi(t,x))\\ \cdot\frac{\partial^2V(t,x)}{\partial x^2}D(t,x,\phi(t,x))\le-\gamma(\|x\|), \tag{4.58}$$

$$\phi(t,0)=0, \tag{4.59}$$

$$L(t,x,\phi(t,x))+\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}F(t,x,\phi(t,x))\\ +\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,\phi(t,x))\frac{\partial^2V(t,x)}{\partial x^2}D(t,x,\phi(t,x))=0, \tag{4.60}$$

$$L(t,x,u)+\frac{\partial V(t,x)}{\partial t}+\frac{\partial V(t,x)}{\partial x}F(t,x,u)\\ +\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u)\frac{\partial^2V(t,x)}{\partial x^2}D(t,x,u)\ge0,\quad(t,x,u)\in[t_0,\infty)\times\mathbb{R}^n\times\mathbb{R}^m. \tag{4.61}$$

Then, with the feedback control $u=\phi(t,x)$, the closed-loop system given by (4.55) is globally uniformly asymptotically stable in probability and $J(t_0,x_0,\phi(\cdot,\cdot))=V(t_0,x_0)$ for all $(t_0,x_0)\in[0,\infty)\times\mathcal{D}_0$. In addition, if $(t_0,x_0)\in[0,\infty)\times\mathbb{R}^n$, then the feedback control $u(\cdot)=\phi(\cdot,x(\cdot))$ minimizes $J(x_0,u(\cdot))$ in the sense that

$$J(t_0,x_0,\phi(\cdot,\cdot))=\min_{u(\cdot)\in\mathcal{S}(t_0,x(t_0))}J(t_0,x_0,u(\cdot)). \tag{4.62}$$

Proof: The proof is a direct consequence of Theorem 4.3 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $F_1(x_1, x_2, u) = F_1(x_2, x_1, u) = F(t, x, u)$, $F_2(x_1, x_2, u) = 1$, $D_1(x_1, x_2, u) = D_1(x_2, x_1, u) = D(t, x, u)$, $D_2(x_1, x_2, u) = 0$, $\phi(x_1, x_2) = \phi(x_2, x_1) = \phi(t, x)$, and $V(x_1, x_2) = V(x_2, x_1) = V(t, x)$. □

Note that (4.60) and (4.61) give the stochastic Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u\in\mathcal{S}(t_0,x(t_0))}\left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x}F(t,x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u)\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x,u)\right], \qquad (t,x)\in[t_0,\infty)\times\mathbb{R}^n, \tag{4.63}$$

which characterizes the optimal control

$$\phi(t,x) = \underset{u\in\mathcal{S}(t_0,x(t_0))}{\arg\min}\left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x}F(t,x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u)\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x,u)\right] \tag{4.64}$$

for time-varying stochastic systems on a finite or infinite interval.

4.4. Partial-State Stochastic Stabilization for Affine Dynamical Systems and Connections to the Time-Varying Linear-Quad--ratic Regulator Problem

In this section, we specialize the results of Section 4.3 to nonlinear affine in the control stochastic dynamical systems of the form

$$\mathrm{d}x_1(t) = [f_1(x_1(t),x_2(t)) + G_1(x_1(t),x_2(t))u(t)]\,\mathrm{d}t + D_1(x_1(t),x_2(t))\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t\ge 0, \tag{4.65}$$

$$\mathrm{d}x_2(t) = [f_2(x_1(t),x_2(t)) + G_2(x_1(t),x_2(t))u(t)]\,\mathrm{d}t + D_2(x_1(t),x_2(t))\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{4.66}$$

where, for every $t\ge 0$, $x_1(t)\in\mathcal{H}_{n_1}$ and $x_2(t)\in\mathcal{H}_{n_2}$, $u(t)\in\mathcal{H}_m$, and $f_1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_1}$, $f_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_2}$, $G_1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_1\times m}$, $G_2:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{n_2\times m}$, $D_1:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to$

$\mathbb{R}^{n_1\times d}$, and $D_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2\times d}$ are such that $f_1(0, x_2) = 0$ and $D_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n_2}$; and $F(x_1, x_2, u) \triangleq [\big(f_1(x_1, x_2) + G_1(x_1, x_2)u\big)^{\mathrm{T}}, \big(f_2(x_1, x_2) + G_2(x_1, x_2)u\big)^{\mathrm{T}}]^{\mathrm{T}}$, $D(x_1, x_2, u) \triangleq [D_1^{\mathrm{T}}(x_1, x_2, u), D_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$ satisfy (4.4) and (4.5) uniformly in u. Furthermore, we consider performance integrands $L(x_1, x_2, u)$ of the form

$$L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}R_2(x_1, x_2)u, \qquad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m}, \tag{4.67}$$

where $L_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $L_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1\times m}$, and $R_2(x_1, x_2) \geq N(x_1) > 0$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, so that (4.36) becomes

$$J(x_{10}, x_{20}, u(\cdot)) = \mathbb{E}^{x_0}\Bigg[\int_0^\infty \big[L_1(x_1(t), x_2(t)) + L_2(x_1(t), x_2(t))u(t) + u^{\mathrm{T}}(t)R_2(x_1(t), x_2(t))u(t)\big]\,\mathrm{d}t\Bigg]. \tag{4.68}$$

For the statement of the next result, define

$$f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}, \qquad G(x_1, x_2) \triangleq [G_1^{\mathrm{T}}(x_1, x_2), G_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}},$$

$$D(x_1, x_2) \triangleq [D_1^{\mathrm{T}}(x_1, x_2), D_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}.$$

Theorem 4.4. Consider the controlled nonlinear affine stochastic dynamical system (4.65) and (4.66) with performance measure (4.68). Assume that there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \ \ (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{4.69}$$

$$V'(x_1, x_2)\big[f(x_1, x_2) - \tfrac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \tfrac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\big] + \tfrac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) \leq -\gamma(\|x_1\|), \ \ (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{4.70}$$

$$L_2(0, x_2) = 0, \ \ x_2 \in \mathbb{R}^{n_2}, \tag{4.71}$$

$$0 = L_1(x_1,x_2) + V'(x_1,x_2)f(x_1,x_2) + \tfrac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2)$$
$$-\tfrac{1}{4}\left[V'(x_1,x_2)G(x_1,x_2) + L_2(x_1,x_2)\right] R_2^{-1}(x_1,x_2)\left[V'(x_1,x_2)G(x_1,x_2) + L_2(x_1,x_2)\right]^{\mathrm{T}},$$
$$(x_1,x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad (4.72)$$

Then, with the feedback control

$$u = \phi(x_1,x_2) = -\tfrac{1}{2}R_2^{-1}(x_1,x_2)\big[L_2(x_1,x_2) + V'(x_1,x_2)G(x_1,x_2)\big]^{\mathrm{T}}, \quad (4.73)$$

the closed-loop system

$$\mathrm{d}x_1(t) = \left[f_1(x_1(t),x_2(t)) + G_1(x_1(t),x_2(t))\phi(x_1(t),x_2(t))\right]\mathrm{d}t + D_1(x_1(t),x_2(t))\mathrm{d}w(t),$$
$$x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \quad (4.74)$$
$$\mathrm{d}x_2(t) = \left[f_2(x_1(t),x_2(t)) + G_2(x_1(t),x_2(t))\phi(x_1(t),x_2(t))\right]\mathrm{d}t + D_2(x_1(t),x_2(t))\mathrm{d}w(t),$$
$$x_2(0) \overset{\text{a.s.}}{=} x_{20}, \quad (4.75)$$

is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20} and the performance measure (4.68) is minimized in the sense of (4.44). Finally,

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))) = V(x_{10},x_{20}), \qquad (x_{10},x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \quad (4.76)$$

Proof: The result is a consequence of Theorem 4.3 with $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, $F(x_1,x_2,u) = f(x_1,x_2) + G(x_1,x_2)u$, and $L(x_1,x_2,u) = L_1(x_1,x_2) + L_2(x_1,x_2)u + u^{\mathrm{T}}R_2(x_1,x_2)u$. Specifically, the feedback control law (4.72) follows from (4.53) by setting

$$\frac{\partial}{\partial u}\big[L_1(x_1,x_2) + L_2(x_1,x_2)u + u^{\mathrm{T}}R_2(x_1,x_2)u + V'(x_1,x_2)\big(f(x_1,x_2) + G(x_1,x_2)u\big)$$
$$+\frac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2)\big] = 0. \quad (4.77)$$

Now, with $u = \phi(x_1,x_2)$ given by (4.73), conditions (4.69), (4.70), and (4.72) imply (4.37), (4.38), and (4.40), respectively.

Next, since $V(\cdot,\cdot)$ is two-times continuously differentiable and, by (4.69), $V(0,x_2)$, $x_2 \in \mathbb{R}^{n_2}$, is a local minimum of $V(\cdot,\cdot)$, it follows that $V'(0,x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and hence, it follows from (4.71) and (4.73) that $\phi(0, x_2) = 0$, which implies (4.39). Finally, since

$$
\begin{aligned}
L(x_1, x_2, u) &+ V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)u] + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) \\
&= L(x_1, x_2, u) + V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)u] \\
&\quad + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) - L(x_1, x_2, \phi(x_1, x_2)) \\
&\quad - V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)\phi(x_1, x_2)] - \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) \\
&= [u - \phi(x_1, x_2)]^{\mathrm{T}} R_2(x_1, x_2)[u - \phi(x_1, x_2)] \\
&\geq 0, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \qquad (4.78)
\end{aligned}
$$

condition (4.41) holds. The result now follows as a direct consequence of Theorem 4.3. □

Next, we use Theorem 4.4 to address the classical time-varying, linear-quadratic stochastic optimal control problem. Specifically, consider the linear time-varying stochastic dynamical system

$$
\mathrm{d}x(t) = [A(t)x(t) + B(t)u(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}(t)\mathrm{d}w(t), \qquad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq t_0, \qquad (4.79)
$$

with performance measure

$$
J(t_0, x_0, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} \left[x^{\mathrm{T}}(t)R_1(t)x(t) + u^{\mathrm{T}}(t)R_2(t)u(t)\right]\mathrm{d}t\right], \qquad (4.80)
$$

where, for all $t \geq t_0$, $x(t) \in \mathcal{H}_n$ and $u(t) \in \mathcal{H}_m$, $\sigma : [t_0, \infty) \to \mathbb{R}^d$, $A : [t_0, \infty) \to \mathbb{R}^{n\times n}$, and $B : [t_0, \infty) \to \mathbb{R}^{n\times m}$ are continuous and uniformly bounded, and $R_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and $R_2 : [t_0, \infty) \to \mathbb{R}^{m\times m}$ are continuous, uniformly bounded, and positive definite, and hence, there exist γ, $\mu > 0$ such that $R_1(t) \geq \gamma I_n > 0$ and $R_2(t) \geq \mu I_m > 0$ for all $t \geq t_0$.

Corollary 4.3. Consider the linear time-varying stochastic dynamical system (4.79) with quad- ratic performance measure (4.80) and let $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ be a continuously differentiable, uniformly bounded, positive definite solution of

$$
-\dot{P}(t) = \left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right)^{\mathrm{T}} P(t) + P(t)\left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right) + R_1(t)
$$

$$- P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t), \qquad \lim_{t_f \to \infty} P(t_f) = \overline{P}, \qquad t \in [t_0, \infty), \tag{4.81}$$

where $\overline{P}$ satisfies (4.81). Then, with the feedback control

$$u = \phi(t, x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{4.82}$$

the stochastic dynamical system (4.79) is globally uniformly asymptotically stable in probability and

$$J(t_0, x_0, \phi(\cdot, \cdot)) = x_0^{\mathrm{T}} P(t_0) x_0, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{4.83}$$

Furthermore, the feedback control $u(\cdot) = \phi(\cdot, x(\cdot))$ minimizes (4.80) in the sense of (4.62).

Proof: The result is a consequence of Theorem 4.4 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = A(t)x$, $f_2(x_1, x_2) = 1$, $G_1(x_1, x_2) = G_1(x_2, x_1) = B(t)$, $G_2(x_1, x_2) = 0$, $D_1(x_1, x_2) = D_1(x_2, x_1) = x\sigma^{\mathrm{T}}(t)$, $D_2(x_1, x_2) = 0$, $L_1(x_1, x_2) = L_1(x_2, x_1) = x^{\mathrm{T}} R_1(t) x$, $L_2(x_1, x_2) = 0$, $R_2(x_1, x_2) = R_2(x_2, x_1) = R_2(t)$, $V(x_1, x_2) = V(x_2, x_1) = x^{\mathrm{T}} P(t) x$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2$, and $\gamma(\|x_1\|) = \gamma\|x\|^2$, for some α, β, $\gamma > 0$. Specifically, since $P(\cdot)$ is uniformly bounded and positive definite, there exist constants $\alpha > 0$ and $\beta > 0$ such that $\alpha I_n \le P(t) \le \beta I_n$, $t \ge t_0$, and hence,

$$\alpha\|x\|^2 \le V(t, x) \le \beta\|x\|^2, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{4.84}$$

which verifies (4.69).

Next, (4.82) is a restatement of (4.73). Now, note that, with $\tilde{A}(t) \triangleq A(t) + B(t)K(t)$, $K(t) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)$, and $\tilde{R}(t) \triangleq R_1(t) + P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)$, (4.81) can be equivalently written as

$$-\dot{P}(t) = \left(\tilde{A}(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right)^{\mathrm{T}} P(t) + P(t)\left(\tilde{A}(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right) + \tilde{R}(t),$$
$$\lim_{t_f \to \infty} P(t_f) = \overline{P}, \qquad t \in [t_0, \infty), \tag{4.85}$$

where $\tilde{A}(t)$, $t \ge t_0$, characterizes the closed-loop dynamics of the closed-loop system (4.79) and (4.82) given by

$$\mathrm{d}x(t) = \tilde{A}(t)x(t)\mathrm{d}t + x(t)\sigma^{\mathrm{T}}(t)\mathrm{d}w(t), \qquad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \ge t_0. \tag{4.86}$$

Next, computing the infinitesimal generator $\mathcal{L}V(t,x)$ along the trajectories of the closed-loop system (4.86) gives

$$\begin{aligned}
\mathcal{L}V(t,x) &= x^{\mathrm{T}}\dot{P}(t)x + 2x^{\mathrm{T}}P(t)\tilde{A}(t)x + \|\sigma(t)\|^2 x^{\mathrm{T}}P(t)x \\
&= x^{\mathrm{T}}\left[\dot{P}(t) + \left(\tilde{A}(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right)^{\mathrm{T}} P(t) + P(t)\left(\tilde{A}(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right)\right] x \\
&= -x^{\mathrm{T}}\tilde{R}(t)x, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n \\
&\le -\gamma\|x\|^2, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \qquad (4.87)
\end{aligned}$$

which verifies (4.70).

Finally, it follows from (4.81) that

$$\begin{aligned}
x^{\mathrm{T}}R_1(t)x &+ \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) + \frac{\partial V(t,x)}{\partial t} \\
&+ \frac{\partial V(t,x)}{\partial x}\big[A(t)x + B(t)\phi(t,x)\big] + \frac{1}{2}\mathrm{tr}\ (x\sigma^{\mathrm{T}}(t))^{\mathrm{T}}\frac{\partial^2 V(t,x)}{\partial x^2}(x\sigma^{\mathrm{T}}(t)) \\
&\quad = x^{\mathrm{T}}\Big[\dot{P}(t) + \left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right)^{\mathrm{T}} P(t) + P(t)\left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right) \\
&\qquad + R_1(t) - P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)\Big]x \\
&\quad = 0, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \qquad (4.88)
\end{aligned}$$

which verifies (4.72). The result now follows as a direct consequence of Theorem 4.4. □

Corollary 4.3 gives sufficient conditions for global uniform asymptotic stability (in probability) and optimality of the linear stochastic dynamical system (4.79) with the state feedback control law (4.82).

4.5. Inverse Optimal Stochastic Control

In this section, we construct state feedback controllers for nonlinear affine in the control stochastic dynamical systems that are predicated on an *inverse optimal control problem* [2, 42, 57, 79, 82].

Theorem 4.5. Consider the nonlinear controlled affine stochastic dynamical system (4.65) and (4.66) with performance measure (4.68). Assume there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \tag{4.89}$$

$$V'(x_1, x_2)\Big[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)$$
$$\cdot G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\Big] + \frac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) \leq -\gamma(\|x_1\|), \tag{4.90}$$

$$L_2(0, x_2) = 0. \tag{4.91}$$

Then, with the feedback control

$$u = \phi(x_1, x_2) = -\frac{1}{2}R_2^{-1}(x_1, x_2)\big[L_2(x_1, x_2) + V'(x_1, x_2)G(x_1, x_2)\big]^{\mathrm{T}}, \tag{4.92}$$

the closed-loop system given by (4.74) and (4.75) is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20} and the performance functional (4.68), with

$$L_1(x_1, x_2) = \phi^{\mathrm{T}}(x_1, x_2)R_2(x_1, x_2)\phi(x_1, x_2) - V'(x_1, x_2)f(x_1, x_2)$$
$$-\frac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2), \tag{4.93}$$

is minimized in the sense of (4.44). Finally, (4.43) holds.

Proof: The proof is identical to the proof of Corollary 4.4. □

Next, we specialize Theorem 4.5 to linear time-varying stochastic systems controlled by nonlinear controllers that minimize a polynomial cost functional generalizing the results of [69] and [45] to the stochastic setting. Specifically, consider the linear time-varying stochastic dynamical system

$$\mathrm{d}x(t) = [A(t)x(t) + B(t)u(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}(t)\mathrm{d}w(t), \qquad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq t_0, \tag{4.94}$$

where, for all $t \geq t_0$, $x(t) \in \mathcal{H}_n$, $u(t) \in \mathcal{H}_m$, and $\sigma : [t_0, \infty) \to \mathbb{R}^d$, $A : [t_0, \infty) \to \mathbb{R}^{n\times n}$, and $B : [t_0, \infty) \to \mathbb{R}^{n\times m}$ are continuous and uniformly bounded. For the following result, let

$R_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}$, $R_2 : [t_0, \infty) \to \mathbb{R}^{m\times m}$, and $\hat{R}_q : [t_0, \infty) \to \mathbb{R}^{n\times n}$, $q = 2, \ldots, r$, where r is a positive integer, be continuous, uniformly bounded, and positive definite matrices, that is, there exist γ, μ, $\hat{\mu}_q > 0$, $q = 2, \ldots, r$, such that $R_1(t) \geq \gamma I_n > 0$, $R_2(t) \geq \mu I_m > 0$, and $\hat{R}_q(t) \geq \hat{\mu}_q I_m > 0$, for all $t \geq t_0$. Furthermore, we consider performance integrands in (4.56) of the form

$$L(t, x, u) = L_1(t, x) + L_2(t, x)u + u^{\mathrm{T}} R_2(t, x)u, \qquad (t, x, u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m, \quad (4.95)$$

where $L_1 : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $L_2 : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{1\times m}$, and $R_2(t, x) \geq N(x) > 0$, $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$, so that (4.56) becomes

$$J(t_0, x_0, u(\cdot)) = \mathbb{E}^{x_0} \left[\int_{t_0}^{\infty} \left[L_1(t, x(t)) + L_2(t, x(t))u(t) + u^{\mathrm{T}}(t) R_2(t, x(t))u(t) \right] \mathrm{d}t \right]. \quad (4.96)$$

Corollary 4.4. Consider the linear controlled time-varying stochastic dynamical system (4.94), where $u(\cdot)$ is admissible. Assume that there exist a uniformly bounded, continuously differentiable, positive definite $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and continuously differentiable, uniformly bounded, nonnegative definite $M_q : [t_0, \infty) \to \mathbb{R}^{n\times n}$, $q = 2, \ldots, r$, such that

$$-\dot{P}(t) = \left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n \right)^{\mathrm{T}} P(t) + P(t) \left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n \right) + R_1(t) - P(t)S(t)P(t),$$
$$\lim_{t_f\to\infty} P(t_f) = \overline{P}, \quad t \in [t_0, \infty), \qquad (4.97)$$

$$-\dot{M}_q(t) = \left(A(t) + \frac{1}{2}(2q-1)\|\sigma(t)\|^2 I_n - S(t)P(t) \right)^{\mathrm{T}} M_q(t)$$
$$+ M_q(t) \left(A(t) + \frac{1}{2}(2q-1)\|\sigma(t)\|^2 I_n - S(t)P(t) \right) + \hat{R}_q(t),$$
$$\lim_{t_f\to\infty} M_q(t_f) = \overline{M}_q, \qquad q = 2, \ldots, r, \qquad t \in [t_0, \infty), \qquad (4.98)$$

where $S(t) \triangleq B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)$ and $\overline{P}$ and $\overline{M}_q$ satisfy (4.97) and (4.98), respectively. Then the zero solution $x(t) \equiv 0$ of the closed-loop system

$$\mathrm{d}x(t) = [A(t)x(t) + B(t)\phi(t, x)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}(t)\mathrm{d}w(t), \qquad x(t_0) = x_0 \text{ a.s.}, \qquad t \geq t_0, \quad (4.99)$$

is globally uniformly asymptotically stable in probability with feedback control

$$u = \phi(t, x) = -R_2^{-1}(t)B^{\mathrm{T}}(t) \left(P(t) + \sum_{q=2}^{r} (x^{\mathrm{T}} M_q(t) x)^{q-1} M_q(t) \right) x, \qquad (4.100)$$

and the performance functional (4.96) with $R_2(t,x) = R_2(t)$, $L_2(t,x) = 0$, and

$$L_1(t,x) = x^{\mathrm{T}}\Bigg(R_1(t) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t) + \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}} S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]\Bigg)x, \quad (4.101)$$

is minimized in the sense of (4.62). Finally,

$$J(t_0, x_0, \phi(\cdot,\cdot)) = x_0^{\mathrm{T}}P(t_0)x_0 + \sum_{q=2}^{r}\frac{1}{q}\Big(x_0^{\mathrm{T}}M_q(t_0)x_0\Big)^q, \qquad (t_0,x_0) \in [0,\infty)\times\mathbb{R}^n. \quad (4.102)$$

Proof: The result is a consequence of Theorem 4.5 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1,x_2) = f_1(x_2,x_1) = A(t)x$, $f_2(x_1,x_2) = 1$, $G_1(x_1,x_2) = G_1(x_2,x_1) = B(t)$, $G_2(x_1,x_2) = 0$, $D_1(x_1,x_2) = D_1(x_2,x_1) = x\sigma^{\mathrm{T}}(t)$, $D_2(x_1,x_2) = 0$, $L_1(x_1,x_2) = L_1(x_2,x_1) = L_1(t,x)$, where $L_1(t,x)$ is given by (4.101), $L_2(x_1,x_2) = 0$, $R_2(x_1,x_2) = R_2(x_2,x_1) = R_2(t)$, $V(x_1,x_2) = V(x_2,x_1) = x^{\mathrm{T}}P(t)x + \sum_{q=2}^{r}\frac{1}{q}(x^{\mathrm{T}}M_q(t)x)^q$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2 + \sum_{q=2}^{r}\frac{1}{q}\hat{\beta}_q^q\|x\|^{2q}$, and $\gamma(\|x_1\|) = -\gamma\|x\|^2 - \sum_{q=2}^{r}\hat{\sigma}_q\hat{\beta}_q^{q-1}\|x\|^{2q}$, for some α, β, γ, $\hat{\beta}_q$, and $\hat{\sigma}_q > 0$, $q = 2,\ldots,r$. Specifically, since $P(\cdot)$ and $M_q(\cdot)$ are uniformly bounded and, respectively, positive and nonnegative definite, there exist constants α, β, and $\hat{\beta}_q > 0$, $q = 2,\ldots,r$, such that $\alpha I_n \le P(t) \le \beta I_n$ and $0 \le M_q(t) \le \hat{\beta}_q I_n$, $t \ge t_0$, and hence,

$$\alpha\|x\|^2 \le V(t,x) \le \beta\|x\|^2 + \sum_{q=2}^{r}\frac{1}{q}\hat{\beta}_q^q\|x\|^{2q}, \qquad (t,x) \in [t_0,\infty)\times\mathbb{R}^n, \quad (4.103)$$

which verifies (4.89).

Next, (4.100) is a restatement of (4.92). Now, let $\phi(t,x) = \phi_1(t,x) + \phi_2(t,x)$, where

$$\phi_1(t,x) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x, \quad (4.104)$$

$$\phi_2(t,x) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x. \quad (4.105)$$

Computing the infinitesimal generator $\mathcal{L}V(t,x)$ along the trajectories of the closed-loop system (4.99) gives

$$\mathcal{L}V(t,x) = x^{\mathrm{T}}\Big(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t)\Big)x + 2x^{\mathrm{T}}P(t)B(t)\phi(t,x) + \|\sigma(t)\|^2x^{\mathrm{T}}P(t)x$$

$$+\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\Big[x^{\mathrm{T}}\left(\dot{M}_q(t)+M_q(t)A(t)+A^{\mathrm{T}}(t)M_q(t)\right)x$$

$$+2x^{\mathrm{T}}M_q(t)B(t)\phi(t,x)+(2q-1)\|\sigma(t)\|^2x^{\mathrm{T}}M_q(t)x\Big]$$

$$=x^{\mathrm{T}}\Bigg(\dot{P}(t)+P(t)\left(A(t)+\frac{1}{2}\|\sigma(t)\|^2I_n\right)+\left(A(t)+\frac{1}{2}\|\sigma(t)\|^2I_n\right)^{\mathrm{T}}P(t)$$

$$-P(t)S(t)P(t)\Bigg)x-x^{\mathrm{T}}P(t)S(t)P(t)x+2x^{\mathrm{T}}P(t)B(t)\phi_2(t,x)$$

$$+\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\Big[x^{\mathrm{T}}(\dot{M}_q(t)+M_q(t)(A(t)+\frac{1}{2}(2q-1)\|\sigma(t)\|^2I_n-S(t)P(t))$$

$$+(A(t)+\frac{1}{2}(2q-1)\|\sigma(t)\|^2I_n-S(t)P(t))^{\mathrm{T}}M_q(t))x+2x^{\mathrm{T}}M_q(t)B(t)\phi_2(t,x)\Big],$$

$$(t,x)\in[t_0,\infty)\times\mathbb{R}^n. \qquad (4.106)$$

Now, using (4.97) and (4.98), (4.106) yields

$$\mathcal{L}V(t,x)=-x^{\mathrm{T}}\left(R_1(t)+\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)\right)x-x^{\mathrm{T}}P(t)S(t)P(t)x$$

$$-2x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}}S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x$$

$$-2x^{\mathrm{T}}P(t)S(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x$$

$$\leq -x^{\mathrm{T}}R_1(t)x-x^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)x$$

$$\leq -\gamma\|x\|^2-\sum_{q=2}^{r}(\hat{\beta}_q\|x\|^2)^{q-1}\hat{\sigma}_q\|x\|^2$$

$$\leq -\gamma\|x\|^2-\sum_{q=2}^{r}\hat{\sigma}_q\hat{\beta}_q^{q-1}\|x\|^{2q}, \qquad (t,x)\in[t_0,\infty)\times\mathbb{R}^n, \qquad (4.107)$$

and hence, (4.90) holds.

Finally, note that

$$\phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x)=x^{\mathrm{T}}P(t)S(t)P(t)x+2x^{\mathrm{T}}P(t)S(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x$$

$$+ x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}} S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x, \tag{4.108}$$

which, using the first equality in (4.107), implies

$$\begin{aligned}\mathcal{L}V(t,x) &= -x^{\mathrm{T}}R_1(t)x - x^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)x - \phi(t,x)R_2(t)\phi(t,x) \\ &\quad - x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}} S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x \\ &= -L_1(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x),\end{aligned} \tag{4.109}$$

where $L_1(t,x)$ is given by (4.101), and thus, (4.93) is verified. The result now follows as a direct consequence of Theorem 4.5. □

Finally, we specialize Theorem 4.5 to linear time-varying stochastic systems controlled by nonlinear controllers that minimize a multilinear cost functional. For the following result, recall $x^{[k]} \triangleq x\otimes x\otimes\cdots\otimes x$ and $\overset{q}{\oplus}A \triangleq A\oplus A\oplus\cdots\oplus A$, with x and A appearing k times, where k is a positive integer. Furthermore, recall $\mathcal{N}^{(k,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^k} : \Psi x^{[k]} \geq 0,\ x\in\mathbb{R}^n\}$ and let $\hat{P}_q : [t_0,\infty) \to \mathbb{R}^{1\times n^{2q}}$, $\hat{R}_{2q} : [t_0,\infty) \to \mathbb{R}^{1\times n^{2q}}$, $q = 2,\ldots,r$, where r is a positive integer, and $R_2 : [t_0,\infty) \to \mathbb{R}^{m\times m}$ be continuous and uniformly bounded, $\hat{R}_{2q}(t)$, $\hat{P}_q(t) \in \mathcal{N}^{(2q,n)}$, and $R_2(t) \geq \mu I_m > 0$, for some $\mu > 0$ and for all $t \geq t_0$.

Corollary 4.5. Consider the linear controlled time-varying stochastic dynamical system (4.94), where $u(\cdot)$ is admissible. Assume that there exist a continuously differentiable, uniformly bounded, positive definite $P : [t_0,\infty) \to \mathbb{R}^{n\times n}$ and continuously differentiable, uniformly bounded $\hat{P}_q : [t_0,\infty) \to \mathbb{R}^{1\times n^{2q}}$, $q = 2,\ldots,r$, such that $\hat{P}_q \in \mathcal{N}^{(k,n)}$,

$$-\dot{P}(t) = \left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right)^{\mathrm{T}} P(t) + P(t)\left(A(t) + \frac{1}{2}\|\sigma(t)\|^2 I_n\right) + R_1(t) - P(t)S(t)P(t),$$
$$\lim_{t_f\to\infty} P(t_f) = \overline{P}, \quad t\in[t_0,\infty), \tag{4.110}$$

$$-\dot{\hat{P}}_q(t) = \hat{P}_q(t)\left[\overset{2q}{\oplus}(A(t) + \frac{1}{2}(2q-1)\|\sigma(t)\|^2 I_n - S(t)P(t))\right] + \hat{R}_{2q}(t), \quad \lim_{t_f\to\infty}\hat{P}_q(t_f) = \overline{\hat{P}}_q,$$
$$q = 2,\ldots,r, \qquad t\in[t_0,\infty), \tag{4.111}$$

where $S(t) \triangleq B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)$ and $\overline{P}$ and $\overline{\hat{P}}_q$ satisfy (4.110) and (4.111), respectively. Then the zero solution $x(t) \equiv 0$ of the closed-loop system (4.99) is globally uniformly asymptotically stable in probability with the feedback control law

$$\phi(t,x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)\left(P(t)x + \frac{1}{2}g'^{\mathrm{T}}(t,x)\right), \tag{4.112}$$

where $g(t,x) \triangleq \sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}$, and the performance functional (4.96) with $R_2(t,x) = R_2(t)$, $L_2(t,x) = 0$, and

$$L_1(t,x) = x^{\mathrm{T}}R_1(t)x + \sum_{q=2}^{r} \hat{R}_{2q}(t)x^{[2q]} + \frac{1}{4}g'(t,x)S(t)g'^{\mathrm{T}}(t,x), \tag{4.113}$$

is minimized in the sense of (4.62). Finally,

$$J(t_0,x_0,\phi(\cdot,\cdot)) = x_0^{\mathrm{T}}P(t_0)x_0 + \sum_{q=2}^{r} \hat{P}_q(t_0)x_0^{[2q]}, \qquad (t_0,x_0) \in [0,\infty) \times \mathbb{R}^n. \tag{4.114}$$

Proof: The result is a consequence of Theorem 4.5 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1,x_2) = f_1(x_2,x_1) = A(t)x$, $f_2(x_1,x_2) = 1$, $G_1(x_1,x_2) = G_1(x_2,x_1) = B(t)$, $G_2(x_1,x_2) = 0$, $D_1(x_1,x_2) = D_1(x_2,x_1) = x\sigma^{\mathrm{T}}(t)$, $D_2(x_1,x_2) = 0$, $L_1(x_1,x_2) = L_1(x_2,x_1) = L_1(t,x)$, where $L_1(t,x)$ is given by (4.113), $L_2(x_1,x_2) = 0$, $R_2(x_1,x_2) = R_2(x_2,x_1) = R_2(t)$, $V(x_1,x_2) = V(x_2,x_1) = x^{\mathrm{T}}P(t)x + \sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2$, and $\gamma(\|x_1\|) = -\gamma\|x\|^2$, for some α, β, $\gamma > 0$. Specifically, since $P(\cdot)$ is uniformly bounded and positive definite there exist constants α, $\beta > 0$ such that $\alpha I_n \leq P(t) \leq \beta I_n$. In addition, since $\hat{P}_q(t) \in \mathcal{N}^{(2q,n)}$, $q = 2,\ldots,n$, for all $t \geq t_0$, it follows that

$$\alpha\|x\|^2 \leq V(t,x) \leq \beta\|x\|^2, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \tag{4.115}$$

which verifies (4.89).

Computing the infinitesimal ch4:generator $\mathcal{L}V(t,x)$ along the trajectories of the closed-loop system (4.99) gives

$$\mathcal{L}V(t,x) = x^{\mathrm{T}}\left(\dot{P}(t) + P(t)A(t) + A^{\mathrm{T}}(t)P(t)\right)x + 2x^{\mathrm{T}}P(t)B(t)\phi(t,x)$$

$$
\begin{aligned}
&+\frac{1}{2}\mathrm{tr}\;(x\sigma^{\mathrm{T}}(t))^{\mathrm{T}}2P(t)x\sigma^{\mathrm{T}}(t)+\sum_{q=2}^{r}\dot{\hat{P}}_q(t)x^{[2q]}+g'(t,x)(A(t)x+B(t)\phi(t,x))\\
&+\frac{1}{2}\mathrm{tr}\;(x\sigma^{\mathrm{T}}(t))^{\mathrm{T}}g''(t,x)x\sigma^{\mathrm{T}}(t)\\
=\;& x^{\mathrm{T}}\Bigg(\dot{P}(t)x+P(t)\left(A(t)+\frac{1}{2}\|\sigma(t)\|^2 I_n\right)+\left(A(t)+\frac{1}{2}\|\sigma(t)\|^2 I_n\right)^{\mathrm{T}}P(t)\\
&-P(t)S(t)P(t)\Bigg)x-x^{\mathrm{T}}P(t)S(t)P(t)x-x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x)+\sum_{q=2}^{r}\dot{\hat{P}}_q(t)x^{[2q]}\\
&+g'(t,x)\left[(A(t)-S(t)P(t))x-\frac{1}{2}S(t)g'^{\mathrm{T}}(t,x)\right]+\frac{1}{2}\mathrm{tr}\;(x\sigma^{\mathrm{T}}(t))^{\mathrm{T}}g''(t,x)x\sigma^{\mathrm{T}}(t)
\end{aligned}
\tag{4.116}
$$

for all $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$. Next, noting that

$$
\begin{aligned}
&g'(t,x)(A(t)-S(t)P(t))x+\frac{1}{2}\mathrm{tr}\;(x\sigma^{\mathrm{T}}(t))^{\mathrm{T}}g''(t,x)x\sigma^{\mathrm{T}}(t)\\
&\quad=\frac{\partial}{\partial x}\left[\sum_{q=2}^{r}\hat{P}_q(t)x^{[2q]}\right](A(t)-S(t)P(t))x+\frac{1}{2}x^{\mathrm{T}}\frac{\partial^2}{\partial x^2}\left[\sum_{q=2}^{r}\hat{P}_q(t)x^{[2q]}\right]x\|\sigma(t)\|^2\\
&\quad=\sum_{q=2}^{r}\hat{P}_q(t)\Bigg(\sum_{i_q=1}^{2q}x\otimes\cdots\otimes\overbrace{I_n}^{i_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes x\Bigg)(A(t)-S(t)P(t))x\\
&\qquad+\sum_{q=2}^{r}\frac{1}{2}\|\sigma(t)\|^2\Bigg(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{i_q=1}^{2q}\sum_{j_q=1,j_q\neq i_q}^{2q}x_i\hat{P}_q(t)(x\otimes\cdots\\
&\qquad\cdots\otimes\overbrace{e_i}^{i_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes\overbrace{e_j}^{j_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes x)x_j\Bigg)\\
&\quad=\sum_{q=2}^{r}\hat{P}_q(t)\Bigg(\sum_{i_q=1}^{2q}x\otimes\cdots\otimes\overbrace{(A(t)-S(t)P(t))x}^{i_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes x\Bigg)\\
&\qquad+\sum_{q=2}^{r}\frac{1}{2}\|\sigma(t)\|^2\Bigg(\sum_{i_q=1}^{2q}\sum_{j_q=1,j_q\neq i_q}^{2q}\sum_{i=1}^{n}\sum_{j=1}^{n}\hat{P}_q(t)(x\otimes\cdots\\
&\qquad\cdots\otimes\overbrace{x_ie_i}^{i_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes\overbrace{x_je_j}^{j_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes x)\Bigg)\\
&\quad=\sum_{q=2}^{r}\hat{P}_q(t)\Bigg(\sum_{i_q=1}^{2q}I_n\otimes\cdots\otimes\overbrace{(A(t)-S(t)P(t))}^{i_q^{\mathrm{th}}\text{ entry}}\otimes\cdots\otimes I_n\Bigg)x^{[2q]}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{q=2}^{r}\frac{1}{2}\|\sigma(t)\|^2\Bigg(\sum_{i_q=1}^{2q}\sum_{j_q=1,j_q\neq i_q}^{2q}\hat{P}_q(t)(x\otimes\cdots\\
&\cdots\otimes\overbrace{(\sum_{i=1}^{n}x_ie_i)}^{i_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes\overbrace{(\sum_{j=1}^{n}x_je_j)}^{j_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes x)\Bigg)\\
&=\sum_{q=2}^{r}\hat{P}_q(t)\left(\sum_{i_q=1}^{2q}I_n\otimes\cdots\otimes\overbrace{(A(t)-S(t)P(t))}^{i_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes I_n\right)x^{[2q]}\\
&\quad+\sum_{q=2}^{r}\frac{1}{2}\|\sigma(t)\|^2\hat{P}_q(t)\left(\sum_{i_q=1}^{2q}\sum_{j_q=1,j_q\neq i_q}^{2q}x\otimes\cdots\otimes\overbrace{x}^{i_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes\overbrace{x}^{j_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes x\right)\\
&=\sum_{q=2}^{r}\hat{P}_q(t)\left(\sum_{i_q=1}^{2q}I_n\otimes\cdots\otimes\overbrace{(A(t)-S(t)P(t))}^{i_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes I_n\right)x^{[2q]}\\
&\quad+\sum_{q=2}^{r}\hat{P}_q(t)\left(\sum_{i_q=1}^{2q}I_n\otimes\cdots\otimes\overbrace{\frac{1}{2}(q-1)\|\sigma(t)\|^2I_n}^{i_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes I_n\right)x^{[2q]}\\
&=\sum_{q=2}^{r}\hat{P}_q(t)\left(\sum_{i_q=1}^{2q}I_n\otimes\cdots\otimes\overbrace{((A(t)-S(t)P(t))+\frac{1}{2}(q-1)\|\sigma(t)\|^2I_n)}^{i_q^{\text{th}}\text{ entry}}\otimes\cdots\otimes I_n\right)x^{[2q]}\\
&=\sum_{q=2}^{r}\hat{P}_q(t)\left[\overset{2q}{\oplus}(A(t)+\frac{1}{2}(2q-1)\|\sigma(t)\|^2I_n-S(t)P(t))\right]x^{[2q]},
\end{aligned}
\tag{4.117}
$$

it follows from (4.110), (4.111), and (4.117), that

$$
\begin{aligned}
\mathcal{L}V(t,x)&=-x^{\mathrm{T}}R_1(t)x-x^{\mathrm{T}}P(t)S(t)P(t)x-x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x)\\
&\quad+\sum_{q=2}^{r}\left(\dot{\hat{P}}_q(t)+\hat{P}_q(t)\left[\overset{2q}{\oplus}(A(t)+\frac{1}{2}(2q-1)\|\sigma(t)\|^2I_n-S(t)P(t))\right]\right)x^{[2q]}\\
&\quad-\frac{1}{2}g'(t,x)S(t)g'^{\mathrm{T}}(t,x)\\
&=-x^{\mathrm{T}}R_1(t)x-x^{\mathrm{T}}P(t)S(t)P(t)x-x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x)\\
&\quad-\sum_{q=2}^{r}\hat{R}_{2q}(t)x^{[2q]}-\frac{1}{2}g'(t,x)S(t)g'^{\mathrm{T}}(t,x).
\end{aligned}
\tag{4.118}
$$

Finally, note that

$$
\phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x)=\left(x^{\mathrm{T}}P(t)+\frac{1}{2}g'(t,x)\right)S(t)\left(P(t)x+\frac{1}{2}g'^{\mathrm{T}}(t,x)\right)
$$

$$= x^{\mathrm{T}}P(t)S(t)P(t)x + \frac{1}{4}g'(t,x)S(t)g'^{\mathrm{T}}(t,x) + x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t,x), \tag{4.119}$$

which, using (4.118), implies that

$$\mathcal{L}V(t,x) = -x^{\mathrm{T}}R_1(t)x - \sum_{q=2}^{r}\hat{R}_{2q}(t)x^{[2q]} - \frac{1}{4}g'(t,x)S(t)g'^{\mathrm{T}}(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) \tag{4.120}$$

for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$, and hence, (4.90) holds with $\gamma(\|x\|) = -\gamma\|x\|^2$. In addition, writing (4.120) as

$$\mathcal{L}V(t,x) = -L_1(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x), \tag{4.121}$$

where $L_1(t,x)$ is given by (4.113), and thus, (4.93) is verified. The result now follows as a direct consequence of Theorem 4.5. □

4.6. Illustrative Numerical Examples

In this section, we provide two illustrative numerical examples to highlight the optimal and inverse optimal partial-state asymptotic stabilization framework developed in this chapter.

Example 4.1. (**Optimal Partial Stabilization of a Rigid Spacecraft**). Consider the rigid spacecraft with stochastic disturbances given by

$$\mathrm{d}\omega_1(t) = [I_{23}\omega_2(t)\omega_3(t) - \alpha_1\omega_1(t) + u_1(t)]\mathrm{d}t + \sigma_1\omega_1(t)\mathrm{d}w(t), \quad \omega_1(0) \overset{\text{a.s.}}{=} \omega_{10}, \quad t \geq 0, \tag{4.122}$$

$$\mathrm{d}\omega_2(t) = [I_{31}\omega_3(t)\omega_1(t) - \alpha_2\omega_2(t) + u_2(t)]\mathrm{d}t + \sigma_2\omega_2(t)\mathrm{d}w(t), \qquad \omega_2(0) \overset{\text{a.s.}}{=} \omega_{20}, \tag{4.123}$$

$$\mathrm{d}\omega_3(t) = [I_{12}\omega_1(t)\omega_2(t)]\mathrm{d}t + \sigma_3\omega_3(t)\mathrm{d}w(t), \qquad \omega_3(0) = \omega_{30} \ \text{ a.s.}, \tag{4.124}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, $I_{31} \triangleq (I_3 - I_1)/I_2$, $I_{12} \triangleq (I_1 - I_2)/I_3$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $I_1 > I_2 > I_3 > 0$, $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ reflect dissipation in the ω_1 and ω_2 coordinates of the spacecraft, u_1 and u_2 are the spacecraft control moments, and $w(t)$ is a standard Wiener process. Here, the state-dependent disturbances can be used to capture perturbations in atmospheric drag for low altitude (i.e., < 600 km) satellites from the Earth's residual atmosphere as well as J_2 perturbations due to the nonspherical mass distribution of the Earth and its nonuniform mass density. For details see [33, 62]. For this example, we seek a state feedback controller $u = [u_1, u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1,\, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$J(x_{10}, x_{20}, u(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty \left[x_1^{\mathrm{T}}(t)R_1x_1(t) + u^{\mathrm{T}}(t)u(t)\right]\mathrm{d}t\right], \tag{4.125}$$

where $R_1 > 0$, is minimized in the sense of (4.44), and (4.122)–(4.124) is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20}.

Note that (4.122)–(4.124) with performance measure (4.125) can be cast in the form of (4.65) and (4.66) with performance measure (4.68). In this case, Theorem 4.4 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 2$,

$$f(x_1, x_2) = \tilde{f}(x_1, x_2) - Ax_1, \quad \tilde{f}(x_1, x_2) \triangleq \left[I_{23}\omega_2\omega_3,\ I_{31}\omega_3\omega_1,\ I_{12}\omega_1\omega_2\right]^{\mathrm{T}},$$

$$A \triangleq \begin{bmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\end{bmatrix}^{\mathrm{T}}, \quad G(x_1, x_2) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}^{\mathrm{T}}, \quad D(x_1, x_2) = \begin{bmatrix}\sigma_1\omega_1 & \sigma_2\omega_2 & \sigma_3\omega_3\end{bmatrix}^{\mathrm{T}},$$

$L_1(x_1, x_2) = x_1^{\mathrm{T}}R_1x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = I_2$ to characterize the optimal partially stabilizing controller. Specifically, in this case (4.72) reduces to

$$0 = x_1^{\mathrm{T}}R_1x_1 + V'(x_1, x_2)\tilde{f}(x_1, x_2) - V'(x_1, x_2)Ax_1 + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2)$$
$$- \frac{1}{4}V'(x_1, x_2)G(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{4.126}$$

Now, choosing $V(x_1, x_2) = x_1^{\mathrm{T}}Px_1$, where $P > 0$, it follows from (4.126) that

$$0 = x_1^{\mathrm{T}}R_1x_1 + V'(x_1, x_2)\tilde{f}(x_1, x_2) - 2x_1^{\mathrm{T}}PHx_1 + x_1^{\mathrm{T}}\Sigma P\Sigma x_1 - x_1^{\mathrm{T}}PPx_1, \tag{4.127}$$

where $H \triangleq \begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}$, $\Sigma \triangleq \begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}$, and $V'(x_1, x_2)\tilde{f}(x_1, x_2) = 0$ only if $P = \rho J$, where $\rho > 0$ and $J \triangleq \begin{bmatrix}-I_{31} & 0\\ 0 & I_{23}\end{bmatrix}$. In this case, (4.127) and $P = \rho J$ imply that

$$0 = R_1 - 2\rho J\tilde{H} - \rho^2 J^2, \tag{4.128}$$

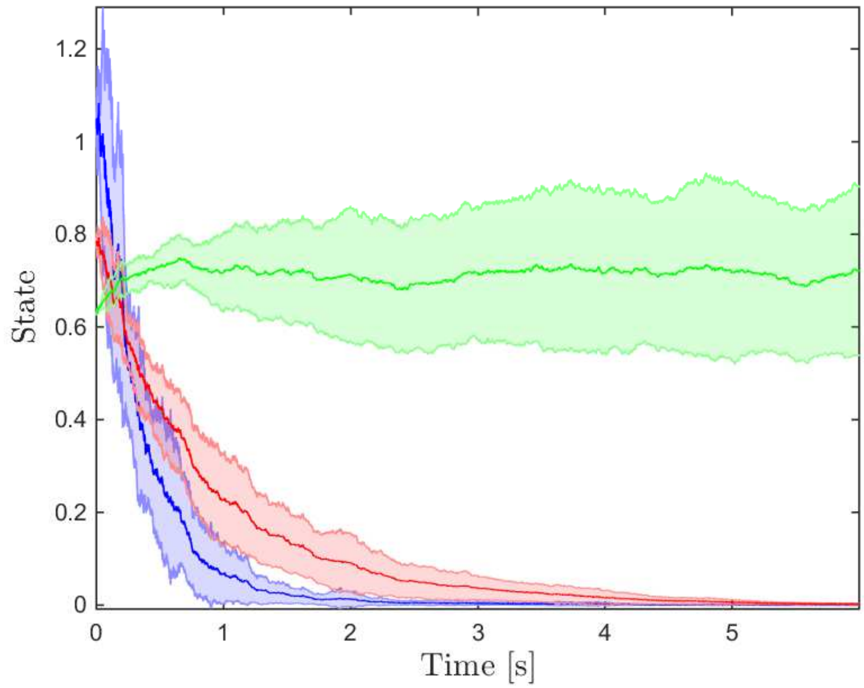

Figure 4.1: A sample average along with the sample standard deviation of the closed-loop system trajectories versus time; $\omega_1(t)$ in blue, $\omega_2(t)$ in red, and $\omega_3(t)$ in green.

where $\tilde{H} = H - \frac{1}{2}\Sigma^2$. Hence, (4.69) holds with $\alpha(\|x_1\|) = \rho\,\lambda_{\min}(J)\|x_1\|^2$ and $\beta(\|x_1\|) = \rho\,\lambda_{\max}(J)\|x_1\|^2$, where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote minimum and maximum eigenvalues, respectively, and (4.70) holds with $\gamma(\|x_1\|) = \lambda_{\min}(R_1)\|x_1\|^2$.

Since all of the conditions of Theorem 4.4 hold, it follows that the feedback control law (4.72) given by

$$\phi(x_1, x_2) = -\frac{1}{2}R_2^{-1}(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2) = -\rho J x_1, \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \quad (4.129)$$

guarantees that the stochastic dynamical system (4.122)–(4.124) is globally asymptotically stable in probability with respect to x_1 uniformly in x_{20} and $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = x_{10}^{\mathrm{T}} P x_{10}$ for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Let $I_1 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_2 = 15\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 10\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = \pi/3\,\mathrm{Hz}$, $\omega_{20} = \pi/4\,\mathrm{Hz}$, $\omega_{30} = \pi/5\,\mathrm{Hz}$, $\alpha_1 = 1.1668\,\mathrm{Hz}$, $\alpha_2 = 0.2\,\mathrm{Hz}$, $\sigma_1 = 1$, $\sigma_2 = 0.4$, $\sigma_3 = 0.1$, and $R_1 = \begin{bmatrix} 5 & 0 \\ 0 & 0.54 \end{bmatrix}\mathrm{Hz}^2$. Figure 4.1 shows the sample average along with the standard deviation of

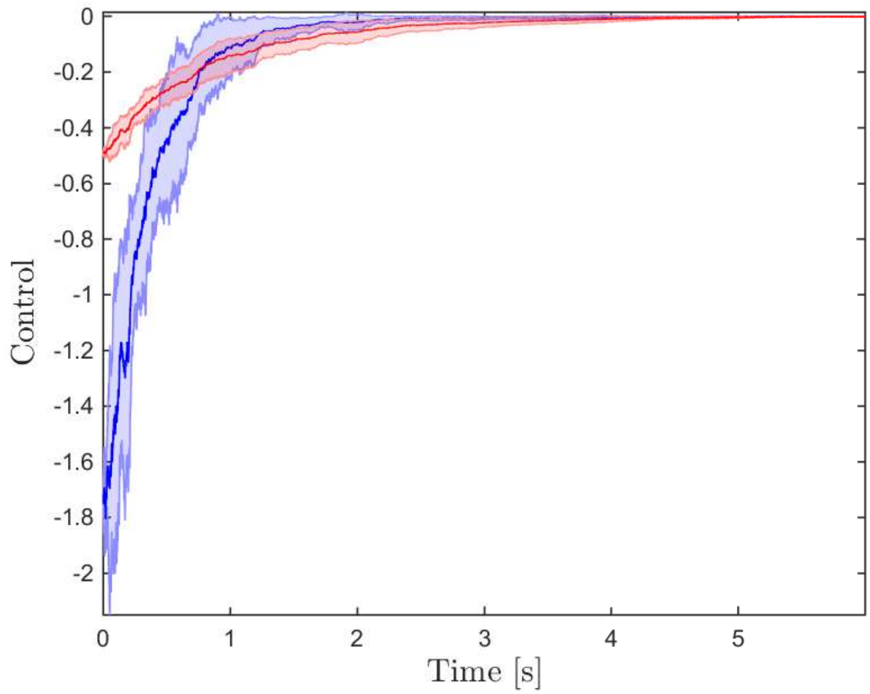

Figure 4.2: A sample average along with the sample standard deviation of the control signal versus time; $u_1(t)$ in blue and $u_2(t)$ in red.

the controlled system state versus time for 20 sample paths for $\rho = 2.5\,\mathrm{Hz}/(\mathrm{N}\cdot\mathrm{m}^2)$. Note that $x_1(t) = [\omega_1(t),\, \omega_2(t)]^{\mathrm{T}} \to 0$ a.s. as $t \to \infty$, whereas $x_2(t) = \omega_3(t)$ does not converge to zero. Figure 4.2 shows the sample average along with the standard deviation of the corresponding control signal versus time Finally, $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = 2.2132\,\mathrm{Hz}^3$. $\triangle$

Example 4.2. (**Thermoacoustic Combustion Model**). In this example, we consider control of thermoacoustic instabilities in combustion processes. Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Due to the inherent coupling between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models [32, 84]. The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develop acoustic pressure and velocity oscillations that can severely affect operating conditions and system performance.

Consider the nonlinear stochastic dynamical system adopted from [32, 45] given by

$$\mathrm{d}q_1(t) = [-\alpha_1 q_1(t) - \beta q_1(t) q_2(t)\cos q_3(t) + u(t)]\mathrm{d}t + \sigma_1 q_1(t)\mathrm{d}w(t),$$
$$q_1(0) \overset{\text{a.s.}}{=} q_{10}, \qquad t \geq 0, \tag{4.130}$$

$$\mathrm{d}q_2(t) = [-\alpha_2 q_2(t) + \beta q_1^2(t)\cos q_3(t) + u(t)]\mathrm{d}t + \sigma_2 q_2(t)\mathrm{d}w(t), \qquad q_2(0) \overset{\text{a.s.}}{=} q_{20} \neq 0, \tag{4.131}$$

$$\mathrm{d}q_3(t) = \left[2\theta_1 - \theta_2 - \beta\left(\frac{q_1^2(t)}{q_2(t)} - 2q_2(t)\right)\sin q_3(t)\right]\mathrm{d}t + \sigma_3 q_1(t) q_2(t)\mathrm{d}w(t), \quad q_3(0) \overset{\text{a.s.}}{=} q_{30}, \tag{4.132}$$

representing a time-averaged, two-mode thermoacoustic combustion model with state dependent stochastic disturbances, where $\alpha_1 > 0$ and $\alpha_2 > 0$ represent decay constants, θ_1 and $\theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = \big((\gamma+1)/8\gamma\big)\omega_1$, where γ denotes the ratio of specific heats and ω_1 is the frequency of the fundamental mode, σ_1, σ_2, and σ_3 are such that $\alpha_1 > \frac{1}{2}\sigma_1^2$ and $\alpha_2 > \frac{1}{2}\sigma_2^2$ and represent augmentation factors of the variance of the state dependent stochastic disturbance, and u is the control input signal. As shown in [32] and [84], only the first two states q_1 and q_2 representing the modal amplitudes of a two-mode thermoacoustic combustion model are relevant in characterizing system instabilities since the third state q_3 represents the phase difference between the two modes [115]. Hence, we require asymptotic stability of $q_1(t)$, $t \geq 0$, and $q_2(t)$, $t \geq 0$, which necessitates partial stabilization.

For this example, we seek a state feedback controller $u = \phi(x_1, x_2)$, where $x_1 = [q_1,\ q_2]^{\mathrm{T}}$ and $x_2 = q_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^\infty \left[x_1^{\mathrm{T}}(t) R_1 x_1(t) + u^2(t)\right]\mathrm{d}t, \tag{4.133}$$

where

$$R_1 = \rho \begin{bmatrix} 2\alpha_1 - \sigma_1^2 + \rho & \rho \\ \rho & 2\alpha_2 - \sigma_2^2 + \rho \end{bmatrix}, \qquad \rho > 0, \tag{4.134}$$

is minimized in the sense of (4.44), and (4.130)–(4.132) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

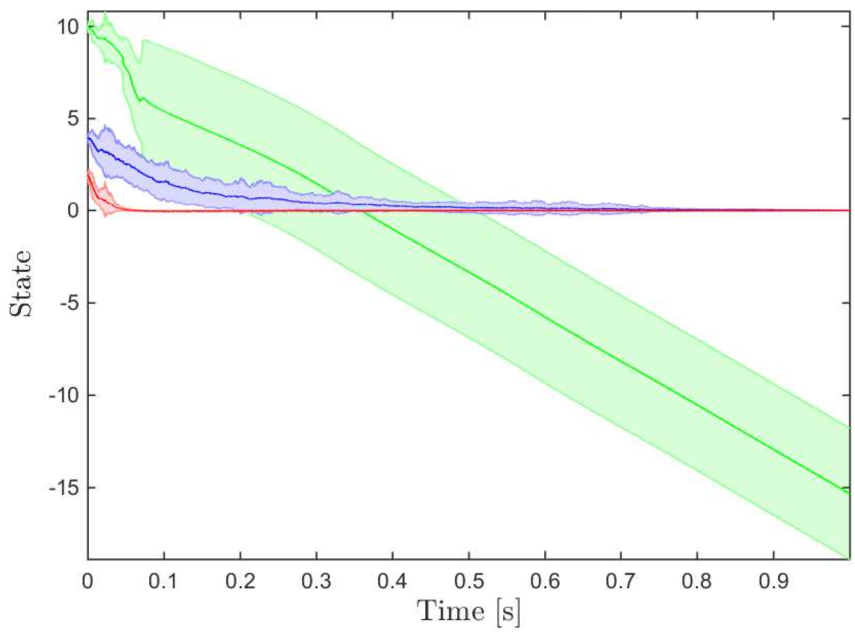

Figure 4.3: A sample average along with the sample standard deviation of the closed-loop system trajectories versus time; $q_1(t)$ in blue, $q_2(t)$ in red, and $q_3(t)$ in green.

Note that (4.130)–(4.132) with performance measure (4.133) can be cast in the form of (4.65) and (4.66) with performance measure (4.68). In this case, Theorem 4.4 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 1$, $f(x_1, x_2) = \big[-\alpha_1 q_1 - \beta q_1 q_2 \cos q_3,\ -\alpha_2 q_2 + \beta q_1^2 \cos q_3,\ 2\theta_1 - \theta_2 - \beta\big(\frac{q_1^2}{q_2} - 2q_2\big)\sin q_3\big]^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^{\mathrm{T}}$, $D(x_1, x_2) = \begin{bmatrix}\sigma_1 q_1 & \sigma_2 q_2 & \sigma_3 q_1 q_2\end{bmatrix}^{\mathrm{T}}$, $L_1(x_1, x_2) = x_1^{\mathrm{T}} R_1 x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = 1$ to characterize the optimal partially stabilizing controller. Specifically, (4.72) reduces to

$$\begin{aligned} 0 = x_1^{\mathrm{T}} R_1 x_1 + V'(x_1, x_2) f(x_1, x_2) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2) V''(x_1, x_2) D(x_1, x_2) \\ - \frac{1}{4} V'(x_1, x_2) G(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{4.135}$$

which implies that $V'(x_1, x_2) = 2\rho\,[q_1,\ q_2,\ 0]$. Furthermore, since $V(0, x_2) = 0$, $x_2 \in \mathbb{R}$, $V(x_1, x_2) = \rho\, x_1^{\mathrm{T}} x_1$, which is positive definite with respect to x_1, and hence, (4.69) holds.

Since all of the conditions of Theorem 4.4 hold, it follows that the feedback control (4.73) given by

$$\phi(x_1, x_2) = -\frac{1}{2} R_2^{-1}(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2)$$

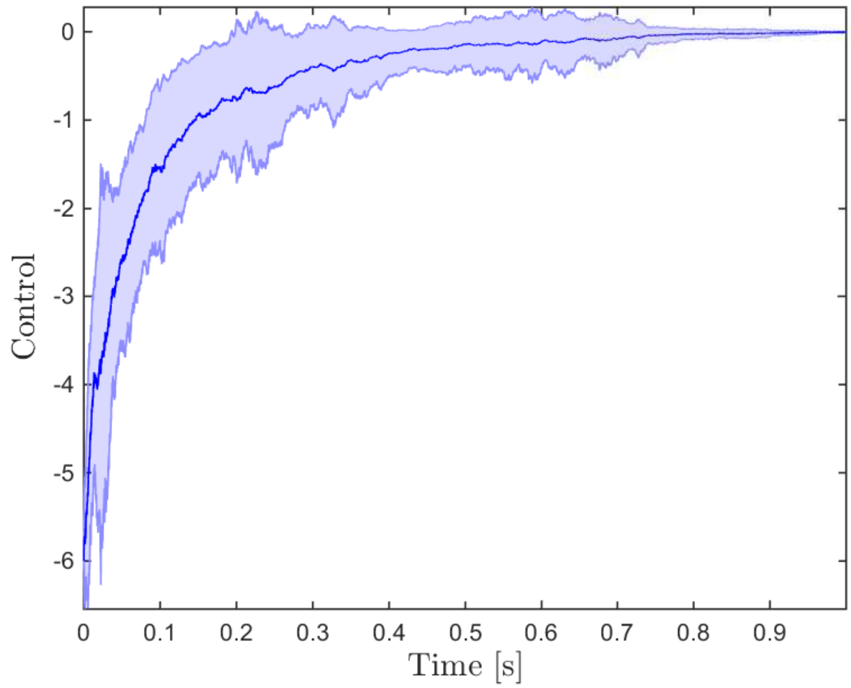

Figure 4.4: A sample average along with the sample standard deviation of the control signal versus time.

$$= -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & 0 \end{bmatrix}^{\mathrm{T}}$$

$$= -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{4.136}$$

guarantees that the dynamical system (4.130)–(4.132) is globally asymptotically stable with respect to x_1 uniformly in x_{20} and $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = \rho\, x_{10}^{\mathrm{T}} x_{10}$ for all $(x_{10}, x_{20}) \in \mathbb{R}^2 \times \mathbb{R}$.

Let $\alpha_1 = 5\,\mathrm{Hz}$, $\alpha_2 = 45\,\mathrm{Hz}$, $\sigma_1 = 2$, $\sigma_2 = 5$, $\sigma_3 = 1$, $\gamma = 1.4$, $\omega_1 = 1\,\mathrm{Hz}$, $\theta_1 = 4\,\mathrm{Hz}$, $\theta_2 = 32\,\mathrm{Hz}$, $\rho = 1\,\mathrm{Hz}$, $q_{10} = 4$, $q_{20} = 2$, and $q_{30} = 10$. Figure 4.3 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figure 4.4 shows the sample average along with the standard deviation of the corresponding control signal versus time for 20 sample paths. Note that $x_1(t) = [q_1(t),\, q_2(t)]^{\mathrm{T}} \overset{\text{a.s.}}{\to} 0$ as $t \to \infty$, whereas $x_2(t) = q_3(t)$ is unstable. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 20\,\mathrm{Hz}$. △

Chapter 5

Stochastic Finite-Time Partial Stability, Partial-State Stabilization, and Finite-Time Optimal Feedback Control

5.1. Introduction

In this chapter, we extend the framework developed in [14] and [45] to address the problem of optimal finite-time stabilization as well as partial-state stabilization for stochastic dynamical systems. The problems of finite-time stochastic stabilization, *optimal* finite-time stochastic stabilization, *optimal* partial-state stochastic stabilization, as well as the combined problem of *optimal finite-time*, *partial-state stochastic stabilization* have not been addressed in the literature. In this chapter, we address these problems by considering a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. In particular, an optimal finite-time, partial-state stochastic stabilization control problem is stated and sufficient stochastic Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the stochastic Hamilton-Jacobi-Bellman equation is clearly shown to be a Lyapunov function for part of the closed-loop system state that guarantees both finite-time partial stability in probability and optimality. In addition, we explore connections of our approach with inverse optimal control [34,42,57,79,82], wherein we parametrize a family of finite-time, partial-state stabilizing stochastic feedback controllers that minimize a derived cost functional. As discussed in Chapter 4, another important application of deterministic partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [28, 45]. We exploit this unification and specialize our results to address the problem of optimal finite-time control for nonlinear time-varying stochastic dynamical systems.

The contents of this chapter are as follows. In Section 5.2, we establish additional notation and several definitions for finite-time, partial-state stability for equilibria of Markov diffusion dynamical systems that have unique solutions in forward time. In Section 5.3, we give a Lyapunov theorem for finite-time, partial stability in probability. Specifically, we present sufficient conditions for finite-time partial stability in probability of nonlinear stochastic dynamical systems using Lyapunov functions that are positive definite with respect to part of the system's state and additionally satisfy a differential inequality involving fractional powers. These results are then specialized to provide sufficient conditions for finite-time stability of nonlinear time-varying stochastic dynamical systems.

In Section 5.4, we consider a nonlinear stochastic system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees finite-time partial stability in probability. We then state a stochastic optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing finite-time partial stability in probability of the closed-loop system. These results are then used to construct optimal finite-time controllers for nonlinear time-varying stochastic dynamical systems. In Section 5.5, we specialize the results developed in Section 5.4 to affine in the control dynamical systems as well as develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the finite-time, partial-state stochastic stabilization problem. In Section 5.6, we provide two illustrative numerical examples that highlight the optimal finite-time, partial-state stochastic stabilization framework.

5.2. Definitions and Mathematical Preliminaries

In this section, we consider nonlinear stochastic autonomous dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x_1(t) = f_1(x_1(t), x_2(t))\mathrm{d}t + D_1(x_1(t), x_2(t))\mathrm{d}w(t), \quad x_1(t_0) \overset{\text{a.s.}}{=} x_{10}, \qquad t \geq t_0, \tag{5.1}$$

$$\mathrm{d}x_2(t) = f_2(x_1(t), x_2(t))\mathrm{d}t + +D_2(x_1(t), x_2(t))\mathrm{d}w(t), \quad x_2(t_0) \overset{\text{a.s.}}{=} x_{20}, \tag{5.2}$$

where, for every $t \geq t_0$, $x_1(t) \in \mathcal{H}_{n_1}$ and $x_2(t) \in \mathcal{H}_{n_2}$ are such that $x(t) \triangleq \left[x_1^{\mathrm{T}}(t),\ x_2^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ is a $\mathcal{F}_t$-measurable random state vector, $x(t_0) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, $w(\cdot)$ is a d-dimensional independent standard Wiener process (i.e., Brownian motion) defined on a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$, $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is continuous in x_1, and $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathbb{R}^{n_1}$, $f_2(x_1, \cdot)$ is continuous in x_2. In addition, the function $D_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1\times d}$ is continuous such that, for every $x_2 \in \mathbb{R}^{n_2}$, $D_1(0, x_2) = 0$, and $D_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2\times d}$ is continuous.

A $\mathbb{R}^{n_1+n_2}$-valued stochastic process $x : [t_0, \tau] \times \Omega \to \mathbb{R}^{n_1+n_2}$ is said to be a *solution* of (5.1) and (5.2) on the interval $[t_0, \tau]$ with initial condition $x(t_0) \overset{\text{a.s.}}{=} x_0$ if $x(\cdot)$ is *progressively measurable* (i.e., $x(\cdot)$ is nonanticipating and measurable in t and ω) with respect to $\{\mathcal{F}_t\}_{t\geq t_0}$, $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}} \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, $D(x_1, x_2) \triangleq [D_1^{\mathrm{T}}(x_1, x_2), D_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}} \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$, and

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s))\mathrm{d}s + \int_{t_0}^{t} D(x(s))\mathrm{d}w(s) \ \text{ a.s.}, \quad t \in [t_0, \tau], \tag{5.3}$$

where the integrals in (5.3) are Itô integrals. As in Chapter 4, we assume that all right maximal pathwise solutions to (5.1) and (5.2) in $(\Omega, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P}^{x_0})$ exist on $[t_0, \infty)$, and hence, we assume (5.1) and (5.2) is forward complete.

Furthermore, we assume that $f : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1+n_2}$ and $D : \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{(n_1+n_2)\times d}$ satisfy the uniform Lipschitz continuity condition, modulo the origin,

$$\|f(x) - f(y)\| + \|D(x) - D(y)\|_{\mathrm{F}} \leq L\|x - y\|, \quad x, y \in \mathbb{R}^{n_1+n_2}\backslash\{0\}, \tag{5.4}$$

and the growth restriction condition $\|f(x)\|^2+\|D(x)\|_{\rm F}^2 \leq L^2(1+\|x\|^2)$, $x \in \mathbb{R}^{(n_1+n_2)}\backslash\{0\}$, for some Lipschitz constant $L > 0$, and hence, since $x(t_0) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ and $x(t_0)$ is independent of $(w(t) - w(t_0)), t \geq t_0$, it follows that there exists a unique, up to equivalence, solution $x \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$ of (5.1) and (5.2) modulo the origin. Specifically, the nonlinear dynamical system given by (5.1) and (5.2) possesses unique solutions in forward time for all initial conditions except possibly at $x_1 \overset{\rm a.s.}{=} 0$ in the following sense. For every $x \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ there exists $\tau_x > 0$ such that, if $x_{\rm I} : [t_0, \tau_1] \times \Omega \to \mathbb{R}^{n_1+n_2}$ and $x_{\rm II} : [t_0, \tau_2] \times \Omega \to \mathbb{R}^{n_1+n_2}$ are two solutions of (5.1) and (5.2) with $x_{\rm I}(0) = x_{\rm II}(0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $x_{\rm I}(t) \overset{\rm a.s.}{=} x_{\rm II}(t)$, $t_0 \leq t \leq \tau_x$. Without loss of generality, we assume that for every (x_1, x_2), τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x = [x_1^{\rm T}, x_2^{\rm T}]^{\rm T} \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, we denote by the measurable map $s^x(\cdot) \triangleq s(\cdot, x_1, x_2)$, corresponding to a unique strongly continuous contraction semigroup, the *trajectories* or the unique *solution curves* of (5.1) and (5.2) on $[0, \tau_x)$ satisfying $s(0, x_1, x_2) = [x_1^{\rm T}, x_2^{\rm T}]^{\rm T}$ and we denote by $s_1^x(\cdot)$ the *partial trajectories* or the unique *solution curves* of (5.1) on $[0, \tau_x)$. Sufficient conditions for forward existence and uniqueness in the absence of the uniform Lipschitz continuity condition and growth restriction condition can be found in [105, 114].

The following definition introduces different notions of stochastic finite-time partial stability.

Definition 5.1. The nonlinear stochastic dynamical system $\mathcal{G}$ given by (5.1) and (5.2) is (*globally*) *stochastic finite-time stable with respect to* x_1 if there exists an operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$, called the *stochastic settling-time operator*, such that the following statements hold:

i) Finite-time, partial-state convergence in probability. For every $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, $s^{x(0)}(t)$ is defined on $[0, T(x_1(0), x_2(0)))$, where $x(0) = [x_1(0)^{\rm T}, x_2(0)^{\rm T}]^{\rm T}$, $s_1^{x(0)}(t) \in \mathcal{H}_{n_1}$ for all $t \in [0, T(x_1(0), x_2(0)))$, and

$$\mathbb{P}^{x_0}\left(\lim_{t\to T(x_1(0),x_2(0))} \|s_1^{x(0)}(t)\| = 0\right) = 1.$$

ii) Partial Lyapunov stability in probability. For every $\varepsilon > 0$, $\rho \in (0,1)$, and $x_2(0) \in \mathcal{H}_{n_2}$, there exists $\delta = \delta(\rho, \varepsilon, x_2(0)) > 0$ such that, for every $x_1(0) \in \mathcal{H}_n^{\mathcal{B}_\delta(0)\backslash\{0\}}$,

$$\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(x_1(0),x_2(0)))} \|s_1^{x(0)}(t)\| \leq \varepsilon\right) \geq 1-\rho.$$

iii) Finiteness of the stochastic settling-time operator. For every $x \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ the stochastic settling-time operator $T(x)$ exists and is finite with probability one, that is, $\mathbb{E}^x\left[T(x)\right] < \infty$.

The nonlinear stochastic dynamical system $\mathcal{G}$ is (*globally*) *stochastic finite-time stable with respect to* x_1 *uniformly in* $x_2(0)$ if $\mathcal{G}$ is stochastic finite-time stable with respect to x_1 and the following statement holds:

iv) Partial uniform Lyapunov stability in probability. For every $\varepsilon > 0$ and $\rho \in (0,1)$ there exists $\delta = \delta(\rho, \varepsilon) > 0$ such that, for every $x_1(0) \in \mathcal{H}_n^{\mathcal{B}_\delta(0)\backslash\{0\}}$,

$$\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(x_1(0),x_2(0)))} \|s_1^{x(0)}(t)\| \leq \varepsilon\right) \geq 1-\rho, \quad x_2(0) \in \mathcal{H}_{n_2}.$$

The nonlinear stochastic dynamical system $\mathcal{G}$ is (*globally*) *strongly stochastic finite-time stable with respect to* x_1 *uniformly in* x_{20} if $\mathcal{G}$ is uniformly stochastic finite-time stable with respect to x_1 and the following statement holds:

v) Finite-time partial uniform convergence in probability. For every $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, $s^{x(0)}(t)$ is defined on $[0, T(x_1(0), x_2(0)))$ and

$$\mathbb{P}^{x_0}\left(\lim_{t\to T(x_1(0),x_2(0))} \|s_1^{x(0)}(t)\| = 0\right) = 1,$$

uniformly in $x_2(0)$ for all $x_2(0) \in \mathcal{H}_{n_2}$.

As noted in Chapter 4, an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying stochastic dynamical system given by

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + D(t, x(t))\mathrm{d}w(t), \qquad x(t_0) \stackrel{\text{a.s.}}{=} x_0, \qquad t \in \mathcal{I}_{x_0,t_0}, \tag{5.5}$$

where, for every $t \in \mathcal{I}_{t_0,x_0}$, $x(t) \in \mathcal{H}_n$ is a $\mathcal{F}_t$-measurable random state vector, $\mathcal{I}_{t_0,x_0} \subseteq [t_0, \infty)$ is the maximal interval of existence of a solution $x(t)$ of (5.5), and $f : \mathcal{I}_{t_0,x_0} \times \mathbb{R}^n \to \mathbb{R}^n$ and $D : \mathcal{I}_{t_0,x_0} \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ are such that, for every $(t, x) \in \mathcal{I}_{t_0,x_0} \times \mathbb{R}^n$, $f(t, 0) = 0$ and $D(t, 0) = 0$, and $f(\cdot, \cdot)$ and $D(\cdot, \cdot)$ are jointly continuous in t and x.

In this chapter, we assume that the nonlinear time-varying stochastic dynamical system (5.5) possesses unique solutions forward in time for all initial conditions except at $x = 0$ and, given $x(t_0) \in \mathcal{H}_n$, we denote by the measurable map $s^{t_0,x(t_0)}(\cdot) \triangleq s(\cdot, t_0, x(t_0))$ the *trajectories* or the unique *solution curves* of (5.5) on $\mathcal{I}_{t_0,x(t_0)}$ satisfying $s(0, t_0, x(t_0)) = x(t_0)$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$ a.s., where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \in \mathcal{I}_{t_0,x_0}$, to the nonlinear time-varying stochastic dynamical system (5.5) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \in \mathcal{T}_{t_0,x_0}$, to the nonlinear autonomous stochastic dynamical system

$$\mathrm{d}x_1(\tau) = f(x_2(\tau), x_1(\tau))\mathrm{d}\tau + D(x_2(\tau), x_1(\tau))\mathrm{d}w(\tau), \quad x_1(0) \stackrel{\text{a.s.}}{=} x_0, \quad \tau \in \mathcal{T}_{t_0,x_0}, \tag{5.6}$$

$$\mathrm{d}x_2(\tau) = \mathrm{d}\tau, \qquad x_2(0) \stackrel{\text{a.s.}}{=} t_0, \tag{5.7}$$

where $\mathcal{T}_{t_0,x_0} \triangleq \{\tau \in \overline{\mathbb{R}}_+ : \tau = t - t_0, t \in \mathcal{I}_{t_0,x_0}\}$. Note that (5.6) and (5.7) are in the same form as the system given by (5.1) and (5.2), and hence, Definition 5.1 applied to (5.6) and (5.7) specializes to the following definition.

Definition 5.2. The nonlinear stochastic dynamical system (5.5) is (*globally*) *stochastic finite-time stable* if there exists an operator $T : [0, \infty) \times \mathcal{H}_n \to \mathcal{H}_1^{[t_0,\infty)}$, called the *stochastic settling-time operator*, such that the following statements hold:

i) Finite-time convergence in probability. For every $(t_0, x(t_0)) \in [0, \infty) \times \mathcal{H}_n$, $s^{t_0,x(t_0)}(t)$ is defined on $[t_0, T(t_0, x(t_0)))$ and

$$\mathbb{P}^{x_0}\left(\lim_{t\to T(t_0,x(t_0))} \|s^{t_0,x(t_0)}(t)\| = 0\right) = 1.$$

ii) *Lyapunov stability in probability.* For every $\varepsilon > 0$, $\rho \in (0,1)$, and $t_0 \in [0,\infty)$, there exists $\delta = \delta(\varepsilon, \rho, t_0) > 0$ such that, for every $x(t_0) \in \mathcal{H}_n^{\mathcal{B}_\delta(0)\backslash\{0\}}$,

$$\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(t_0,x(t_0)))} \|s^{t_0,x(t_0)}(t)\| \leq \varepsilon\right) \geq 1-\rho.$$

iii) *Finiteness of the stochastic settling-time operator.* For every $t \in [t_0,\infty)$ and $x \in \mathcal{H}_n$ the stochastic settling-time operator $T(t,x)$ exists and is finite with probability one, that is, $\mathbb{E}^{t,x}[T(t,x)] < \infty$.

The nonlinear stochastic dynamical system (5.5) is (*globally*) *uniformly stochastic finite-time stable* if (5.5) is (globally) stochastic finite-time stable and the following statement holds:

iv) *Uniform Lyapunov stability in probability.* For every $\varepsilon > 0$ and $\rho \in (0,1)$ there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that, for every $x(t_0) \in \mathcal{H}_n^{\mathcal{B}_\delta(0)\backslash\{0\}}$,

$$\mathbb{P}^{x_0}\left(\sup_{t\in[0,T(t_0,x(t_0)))} \|s^{t_0,x(t_0)}(t)\| \leq \varepsilon\right) \geq 1-\rho, \quad t_0 \in [0,\infty).$$

The nonlinear stochastic dynamical system (5.5) is (*globally*) *strongly uniformly stochastic finite-time stable* if (5.5) is uniformly stochastic finite-time stable and the following statement holds:

v) *Finite-time uniform convergence in probability.* For every $(t_0, x(t_0)) \in [0,\infty) \times \mathcal{H}_n$, $s^{t_0,x(t_0)}(t)$ is defined on $[t_0, T(t_0,x(t_0)))$ and

$$\mathbb{P}^{x_0}\left(\lim_{t\to T(t_0,x(t_0))} \|s^{t_0,x(t_0)}(t)\| = 0\right) = 1,$$

uniformly in t_0 for all $t_0 \in [0,\infty)$.

Remark 5.1. The notion of finite-time stability introduced here is different from the same term discussed in [35]. Specifically, the term finite-time stability discussed in [35] deals with systems whose operation is constrained to a fixed finite interval of time and requires bounds on the system state variables.

5.3. Stochastic Finite-Time Partial Stability Theory

In this section, we address finite-time partial stability in probability for equilibria of Markov diffusion processes that have unique solutions. Sample continuity and uniqueness render the system solutions continuous with respect to the system initial conditions, and hence, the solutions define a time-homogeneous semigroup of Markov kernels. The following proposition shows that if the nonlinear stochastic dynamical system (5.1) and (5.2) is stochastic finite-time stable with respect to x_1, then (5.1) and (5.2) possesses a unique solution $s(\cdot, x_1(0), x_2(0))$ defined on $\overline{\mathbb{R}}_+ \times \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ for every $x_1(0) \in \mathcal{H}_{n_1}$ and, for every $x_2(0) \in \mathcal{H}_{n_2}$,

$$\mathbb{P}^{x_0}\left(\sup_{t \geq T(x_1(0),x_2(0))} \|s_1(t, x_1(0), x_2(0))\| = 0 \right) = 1,$$

where $x(0) \triangleq [x_1(0)^{\mathrm{T}}, x_2(0)^{\mathrm{T}}]^{\mathrm{T}}$ and $T(0, x_2(0)) \triangleq 0$ a.s.

Proposition 5.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (5.1) and (5.2). Assume that $\mathcal{G}$ is globally stochastic finite-time stable with respect to x_1 and let $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ be defined as in Definition 5.1. Then, for every $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, there exists a unique solution $s(t, x_1(0), x_2(0)) = [s_1^{\mathrm{T}}(t, x_1(0), x_2(0)), s_2^{\mathrm{T}}(t, x_1(0), x_2(0))]^{\mathrm{T}}$, $t \geq 0$, to (5.1) and (5.2) defined on $\overline{\mathbb{R}}_+ \times \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ such that $s_1(t, x_1(0), x_2(0)) \in \mathcal{H}_{n_1}$, $t \in [0, T(x_1(0), x_2(0)))$, and such that $\mathbb{P}^{x_0}\left(\sup_{t \geq T(x_1(0),x_2(0))} \|s_1(t, x_1(0), x_2(0))\| = 0\right) = 1$, where $x_0 \triangleq [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$ and $T(0, x_2(0)) \triangleq 0$ a.s.

Proof: It follows from the partial Lyapunov stability in probability of (5.1) and (5.2) with respect to x_1 that $x_1(t) \overset{\text{a.s.}}{\equiv} 0$, $t \geq 0$, is the unique solution of (5.1) satisfying $x_1(0) \overset{\text{a.s.}}{=} 0$ for all $x_2(0) \in \mathcal{H}_{n_2}$. Thus, $s_1(t, 0, x_2(0)) \overset{\text{a.s.}}{=} 0$ for all $t \geq 0$ and $x_2(0) \in \mathcal{H}_{n_2}$. Next, let $T(\cdot, \cdot)$ be as in Definition 5.1, and let $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1}^{\mathbb{R}^{n_1}\backslash\{0\}} \times \mathcal{H}_{n_2}$. Furthermore, define

$$x_1(t) \triangleq \begin{cases} s_1(t, x_1(0), x_2(0)), & 0 \leq t < T(x_1(0), x_2(0)), \\ 0, & t \geq T(x_1(0), x_2(0)). \end{cases} \tag{5.8}$$

Note that by construction, the stochastic differential of $x_1(\cdot)$ is sample continuous (i.e., almost surely continuous) on $\overline{\mathbb{R}}_+ \setminus \{T(x_1(0),\ x_2(0))\}$ and satisfies (5.1) on $\overline{\mathbb{R}}_+ \setminus \{T(x_1(0), x_2(0))\}$. Furthermore, since $f_1(\cdot,\cdot)$ and $D_1(\cdot,\cdot)$ are jointly continuous,

$$\begin{aligned}\lim_{t\to T^-(x_1(0),x_2(0))} \mathrm{d}x_1(t) &\overset{\text{a.s.}}{=} \lim_{t\to T^-(x_1(0),x_2(0))} \big(f_1(x_1(t),x_2(t))\mathrm{d}t + D_1(x_1(t),x_2(t))\mathrm{d}w(t)\big)\\ &\overset{\text{a.s.}}{=} \lim_{t\to T^+(x_1(0),x_2(0))} \big(f_1(x_1(t),x_2(t))\mathrm{d}t + D_1(x_1(t),x_2(t))\mathrm{d}w(t)\big)\\ &\overset{\text{a.s.}}{=} \lim_{t\to T^+(x_1(0),x_2(0))} \mathrm{d}x_1(t),\end{aligned}$$

and hence, $x_1(\cdot)$ has a sample continuous stochastic differential at $T(x_1(0), x_2(0))$ and $x_1(\cdot)$ satisfies (5.1). Hence, it follows from the assumptions on $f_2(\cdot,\cdot)$ and $D_2(\cdot,\cdot)$ that, given $x_1(t)$, $t \geq 0$, there exists $x_2(t)$ such that $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is solution of (5.1) and (5.2) for all $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1}^{\mathbb{R}^{n_1}\setminus\{0\}} \times \mathcal{H}_{n_2}$ and $t \geq 0$.

Given $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, to show uniqueness, up to equivalence, assume $y_1(\cdot)$ satisfies (5.1) for all $t \geq 0$. In this case, $x_1(t) \overset{\text{a.s.}}{=} y_1(t)$ for all $t \in [0, T(x_1(0), x_2(0)))$ by the uniqueness assumption in Section 5.2. In addition, by sample continuity, $x_1(t) \overset{\text{a.s.}}{=} y_1(t)$ at $t = T(x_1(0), x_2(0))$, and hence, $x_1(t) \overset{\text{a.s.}}{=} y_1(t)$ for all $t \in [0, T(x_1(0), x_2(0))]$, which implies that $y_1(x_1(0), x_2(0))) \overset{\text{a.s.}}{=} 0$. Now, partial Lyapunov stability in probability with respect to x_1 implies that $y_1(t) \overset{\text{a.s.}}{=} 0$ for $t > T(x_1(0), x_2(0))$, which proves uniqueness of $x_1(\cdot)$. Hence, uniqueness of $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ immediately follows from the assumptions in Section 5.2. This proves the result. □

It follows from Proposition 5.1 and the assumptions on $f_2(\cdot,\cdot)$ and $D_2(\cdot,\cdot)$ that if the nonlinear stochastic dynamical system (5.1) and (5.2) is stochastic finite-time stable with respect to x_1, then it defines a *global semiflow* on $\mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$; that is, the solution curve $s(\cdot,\cdot,\cdot)$ of (5.1) and (5.2) satisfies the consistency property $s(0, x_1, x_2) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ and the semigroup property $s(t, s_1(\tau, x_1, x_2), s_2(\tau, x_1, x_2)) \overset{\text{a.s.}}{=} s(t + \tau, x_1, x_2)$ for every $(x_1, x_2) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ and $t,\ \tau \in \overline{\mathbb{R}}_+$. Furthermore, $s(\cdot,\cdot,\cdot)$ satisfies $s_1(T(x_1(0), x_2(0)) + t_1, x_1(0), x_2(0)) \overset{\text{a.s.}}{=} 0$ for all $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ and $t_1 \geq 0$. It is easy to see from Definition 5.1 that

$$T(x_1(0), x_2(0)) = \inf\{t \in \overline{\mathbb{R}}_+ : s_1(t, x_1(0), x_2(0)) = 0\}, \quad (x_1(0), x_2(0)) \in \mathcal{H}_{n_1}^{\mathbb{R}^{n_1}\setminus\{0\}} \times \mathcal{H}_{n_2}.$$

In general, stochastic finite-time partial stability does not imply that the stochastic settling-time operator $T(\cdot,\cdot)$ is sample continuous. The following proposition shows that the stochastic settling-time operator $T(\cdot,\cdot)$ of a stochastic finite-time partially stable system is jointly sample continuous on $\mathcal{H}_{n_1}\times\mathcal{H}_{n_2}$ if and only if it is sample continuous at $(0,\cdot)$.

Proposition 5.2. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (5.1) and (5.2). Assume $\mathcal{G}$ is stochastic finite-time stable with respect to x_1. Let $T:\mathcal{H}_{n_1}\times\mathcal{H}_{n_2}\to\mathcal{H}_1^{[0,\infty)}$ be the stochastic settling-time operator. Then the following statements hold:

i) If $t_1\geq 0$ and $(x_1(0),x_2(0))\in\mathcal{H}_{n_1}\times\mathcal{H}_{n_2}$, then

$$T(s_1(t_1,x_1(0),x_2(0)),s_2(t_1,x_1(0),x_2(0)))\stackrel{\text{a.s.}}{=}\max\{T(x_1(0),x_2(0))-t_1,0\}. \tag{5.9}$$

ii) $T(\cdot,\cdot)$ is jointly sample continuous on $\mathcal{H}_{n_1}\times\mathcal{H}_{n_2}$ if and only if $T(\cdot,\cdot)$ is jointly sample continuous at $(0,x_2)$, $x_2\in\mathcal{H}_{n_2}$.

Proof: *i*) It follows from Definition 5.1 that

$$T(x_1(0),x_2(0))=\inf\{t\in\mathbb{R}_+: s_1(t,x_1(0),x_2(0))=0\} \tag{5.10}$$

for all $(x_1(0),x_2(0))\in\mathcal{H}_{n_1}^{\mathbb{R}^{n_1}\backslash\{0\}}\times\mathcal{H}_{n_2}$. Hence, $T(s_1(t_1,x_1(0),x_2(0)),s_2(t_1,x_1(0),x_2(0)))=\inf\{t_2\in\mathbb{R}_+: s_1(t_2,s_1(t_1,x_1(0),x_2(0)),s_2(t_1,x_1(0),x_2(0)))=0\}$. Now, for $0\leq t_1\leq T(x_1(0),x_2(0))$, the semigroup property and (5.10) imply that

$$\begin{aligned}T(s_1(t_1,&x_1(0),x_2(0)),s_2(t_1,x_1(0),x_2(0)))\\&=\inf\{t_2\in\mathbb{R}_+: s_1(t_2,s_1(t_1,x_1(0),x_2(0)),s_2(t_1,x_1(0),x_2(0)))=0\}\\&\stackrel{\text{a.s.}}{=}\inf\{t_2\in\mathbb{R}_+: s_1(t_1+t_2,x_1(0),x_2(0))=0\}\\&\stackrel{\text{a.s.}}{=}T(x_1(0),x_2(0))-t_1.\end{aligned}$$

Alternatively, for $0\leq T(x_1(0),x_2(0))\leq t_1$, $T(s_1(t_1,x_1(0),x_2(0)),s_2(t_1,x_1(0),x_2(0)))\stackrel{\text{a.s.}}{=}0$, which proves (5.9).

ii) Necessity is immediate. To prove sufficiency, suppose that $T(\cdot,\cdot)$ is jointly sample continuous at $(0, x_2)$, $x_2 \in \mathcal{H}_{n_2}$. Let $(x_1, x_2) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ and consider the sequences $\{x_{1n}\}_{n=1}^{\infty} \in \mathcal{H}_{n_1}$ converging pointwise to x_1 and $\{x_{2n}\}_{n=1}^{\infty} \in \mathcal{H}_{n_2}$ converging pointwise to x_2. Let $\tau^- = \liminf_{n\to\infty} T(x_{1n}, x_{2n})$ and $\tau^+ = \limsup_{n\to\infty} T(x_{1n}, x_{2n})$ be pointwise limits. Note that τ^-, $\tau^+ \in \mathcal{H}_1^{\mathbb{R}_+}$ and $\tau^- \overset{\text{a.s.}}{\leq} \tau^+$.

Next, let $\{x_{1n_m}\}_{m=0}^{\infty} \in \mathcal{H}_{n_1}$ be a subsequence of $\{x_{1n}\}$ and $\{x_{2n_m}\}_{m=0}^{\infty} \in \mathcal{H}_{n_2}$ be a subsequence of $\{x_{2n}\}$ such that $T(x_{1n_m}, x_{2n_m}) \overset{\text{a.s.}}{\to} \tau^+$ as $m \to \infty$. The sequence $\{(T(x_1, x_2), x_{1n_m}, x_{2n_m})\}_{m=1}^{\infty}$ converges in $\mathcal{H}_1^{\overline{\mathbb{R}}_+} \times \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$ to $(T(x_1, x_2), x_1, x_2)$ almost surely as $m \to \infty$. Since $s_1(T(x_1, x_2) + t_1, x_1, x_2) \overset{\text{a.s.}}{=} 0$ for all $t_1 \geq 0$ and since all solutions to (5.1) and (5.2) are sample continuous in their initial conditions [6, Thm. 7.3.1], it follows that $s_1(T(x_1, x_2), x_{1n_m}, x_{2n_m}) \overset{\text{a.s.}}{\to} s_1(T(x_1, x_2), x_1, x_2) \overset{\text{a.s.}}{=} 0$ as $m \to \infty$. Thus, since $T(0, x_2)$ is sample continuous for all $x_2 \in \mathcal{H}_{n_2}$, it follows that

$$T(s_1(T(x_1, x_2), x_{1n_m}, x_{2n_m}), s_2(T(x_1, x_2), x_{1n_m}, x_{2n_m})) \overset{\text{a.s.}}{\to} T(0, s_2(T(x_1, x_2), x_1, x_2)) \overset{\text{a.s.}}{=} 0. \tag{5.11}$$

Now, with $t_1 = T(x_1, x_2)$, $x_1(0) = x_{1n_m}$, and $x_2(0) = x_{2n_m}$, it follows from (5.9) and (5.11) that $T(s_1(T(x_1, x_2), x_{1n_m}, x_{2n_m}), s_2(T(x_1, x_2), x_{1n_m}, x_{2n_m})) \overset{\text{a.s.}}{=} \max\{T(x_{1n_m}, x_{2n_m}) - T(x_1, x_2), 0\}$ and $\max\{T(x_{1n_m}, x_{2n_m}) - T(x_1, x_2), 0\} \overset{\text{a.s.}}{\to} 0$ as $m \to \infty$. Thus, $\max\{\tau^+ - T(x_1, x_2), 0\} \overset{\text{a.s.}}{=} 0$, which implies that $\tau^+ \overset{\text{a.s.}}{\leq} T(x_1, x_2)$.

Finally, let $\{x_{1n_k}\}_{k=0}^{\infty} \in \mathcal{H}_{n_1}$ be a subsequence of $\{x_{1n}\}$ and $\{x_{2n_k}\}_{k=0}^{\infty} \in \mathcal{H}_{n_2}$ be a subsequence of $\{x_{2n}\}$ such that $T(x_{1n_k}, x_{2n_k}) \overset{\text{a.s.}}{\to} \tau^-$ as $k \to \infty$. It follows from $\tau^- \overset{\text{a.s.}}{\leq} \tau^+$ and $\tau^+ \overset{\text{a.s.}}{\leq} T(x_1, x_2)$ that $\tau^- \in \mathcal{H}_1^{\mathbb{R}_+}$, and hence, the sequence $\{(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k})\}_{k=1}^{\infty}$ converges pointwise to (τ^-, x_1, x_2) as $k \to \infty$. Since $s_1(\cdot,\cdot,\cdot)$ is jointly sample continuous, it follows that $s_1(T(x_{1n_k},\ x_{2n_k}), x_{1n_k}, x_{2n_k}) \overset{\text{a.s.}}{\to} s_1(\tau^-, x_1, x_2)$ as $k \to \infty$. Now, since $s_1(T(x_1, x_2)+t_1, x_1, x_2) \overset{\text{a.s.}}{=} 0$ for all $t_1 \geq 0$, $s_1(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k}) \overset{\text{a.s.}}{=} 0$ for each k. Hence, $s_1(\tau^-, x_1, x_2) \overset{\text{a.s.}}{=} 0$ and, by the definition of the settling-time operator, $T(x_1, x_2) \overset{\text{a.s.}}{\leq} \tau^-$. Now, it follows from $\tau^- \overset{\text{a.s.}}{\leq} \tau^+$, $\tau^+ \overset{\text{a.s.}}{\leq} T(x_1, x_2)$, and $T(x_1, x_2) \overset{\text{a.s.}}{\leq} \tau^-$ that $\tau^- \overset{\text{a.s.}}{=} T(x_1, x_2) \overset{\text{a.s.}}{=} \tau^+$, and hence, $T(x_{1n}, x_{2n}) \stackrel{\text{a.s.}}{\to} T(x_1, x_2)$ as $n \to \infty$, which proves that $T(\cdot,\cdot)$ is jointly sample continuous on $\mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$. □

Next, we present sufficient conditions for stochastic finite-time partial stability using a Lyapunov function involving a scalar differential inequality.

Theorem 5.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (5.1) and (5.2). Then the following statements hold:

i) If there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(\|x_2(t)\|)$ is $\mathcal{F}_t$-submartingale, and, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$V(0, x_2) = 0, \tag{5.12}$$

$$\alpha(\|x_1\|) \le V(x_1, x_2), \tag{5.13}$$

$$V'(x_1,x_2)f(x_1,x_2) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2) \le -k(\|x_2\|)r(V(x_1,x_2)), \tag{5.14}$$

$$\int_0^{\varepsilon} \frac{\mathrm{d}v}{r(v)} < \infty, \quad \varepsilon \in [0,\infty), \tag{5.15}$$

$$r'(v) > 0, \qquad v > 0, \tag{5.16}$$

then $\mathcal{G}$ is globally stochastic finite-time stable with respect to x_1. Moreover, there exists a settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_1(0), x_2(0))] \le q^{-1}\left(\int_0^{V(x_{10},x_{20})} \frac{\mathrm{d}v}{r(v)}\right), \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{5.17}$$

where $q : [0,\infty) \to [0,\infty)$ is continuously differentiable and satisfies

$$\dot{q}(t) = \mathbb{E}^{x_0}[k(\|x_2(t)\|)], \qquad q(0) = 0, \qquad t \ge 0. \tag{5.18}$$

ii) If there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(\|x_2(t)\|)$ is $\mathcal{F}_t$-submartingale, a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$, such that (5.13)–(5.16) hold, and

$$V(x_1, x_2) \le \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{5.19}$$

then $\mathcal{G}$ is globally stochastic finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.17) holds.

iii) If there exist a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.13)–(5.16), and (5.19) hold with $k(\|x_2\|) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$, then $\mathcal{G}$ is globally strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_1(0), x_2(0))] \leq \int_0^{V(x_{10},x_{20})} \frac{\mathrm{d}v}{kr(v)}, \quad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{5.20}$$

Proof. *i*) Let $x_1 \in \mathbb{R}^{n_1}$, $x_{20} \in \mathbb{R}^{n_2}$, $\varepsilon > 0$, and $\rho > 0$, and define $\mathcal{D}_{\varepsilon,\rho} \triangleq \{x_1 \in \mathcal{B}_\varepsilon(0) : V(x_1, x_{20}) < \alpha(\varepsilon)\rho\}$. Since $V(\cdot,\cdot)$ is continuous and $V(0, x_2) = 0$, it follows that $\mathcal{D}_{\varepsilon,\rho}$ is nonempty and there exists $\delta = \delta(\varepsilon, \rho, x_{20}) > 0$ such that $V(x_1, x_{20}) < \alpha(\varepsilon)\rho$, $x_1 \in \mathcal{B}_\delta(0)$. Hence, $\mathcal{B}_\delta(0) \subseteq \mathcal{D}_{\varepsilon,\rho}$. Next, it follows from (5.14) that $V(x_1(t), x_2(t))$ is a (positive) supermartingale [67, Lemma 5.4], and hence, for every $x_1(0) \in \mathcal{H}_{n_1}^{\mathcal{B}_\delta(0)} \subseteq \mathcal{H}_{n_1}^{\mathcal{D}_\rho}$, it follows from (5.13), with $\alpha(\cdot) \in \mathcal{K}_\infty$, and the extended version of the Markov inequality for monotonically increasing functions [41, p. 193] that

$$\begin{aligned}
\mathbb{P}^{x_0}\left(\sup_{t\geq 0} \|x_1(t)\| \geq \varepsilon\right) &\leq \sup_{t\geq 0} \frac{\mathbb{E}^{x_0}[\alpha(\|x_1(t)\|)]}{\alpha(\varepsilon)} \\
&\leq \sup_{t\geq 0} \frac{\mathbb{E}^{x_0}[V(x_1(t), x_2(t))]}{\alpha(\varepsilon)} \\
&\leq \frac{\mathbb{E}^{x_0}[V(x_1(0), x_2(0))]}{\alpha(\varepsilon)} \\
&\leq \rho,
\end{aligned}$$

which proves partial Lyapunov stability in probability with respect to x_1.

To prove global partial asymptotic stability in probability, it follows from (5.14) and [75, Corollary 4.2] that $\lim_{t\to\infty} k(\|x_2(t)\|)r(V(x_1(t), x_2(t)) \overset{\text{a.s.}}{=} 0$. Since $k(\|x_2(t)\|)$ is $\mathcal{F}_t$-submartingale, it follows that $\lim_{t\to\infty} r(V(x_1(t), x_2(t)) \overset{\text{a.s.}}{=} 0$, which, since $r : \mathbb{R}_+ \to \mathbb{R}_+$, further implies that $\lim_{t\to\infty} V(x_1(t), x_2(t)) \overset{\text{a.s.}}{=} 0$. Now, it follows from (5.13) that

$$\lim_{t\to\infty} \alpha(\|x_1(t)\|) \leq \lim_{t\to\infty} V(x_1(t), x_2(t)) \overset{\text{a.s.}}{=} 0,$$

which implies $\mathbb{P}^{x_0}\left(\lim_{t\to\infty} \|x_1(t)\| = 0\right) = 1$. Hence, $\mathcal{G}$ is globally partially asymptotically stable in probability and the stochastic settling-time operator $T(x_1(0), x_2(0)) \leq \infty$ almost surely [29].

Next, we show that $T(x_1(0), x_2(0))$ is finite with probability one and satisfies (5.17), and hence, $\mathbb{E}^{x_0}[T(x_1(0), x_2(0))] < \infty$. Define $T_0 \triangleq T(x_1(0), x_2(0))$, $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$, and $\alpha(V) \triangleq \int_0^V \frac{\mathrm{d}v}{r(v)}$, $V \in \overline{\mathbb{R}}_+$. Now, using Itô's (chain rule) formula the stochastic differential of $V(x(t))$ along the system trajectories $x(t)$, $t \geq 0$, is given by

$$\mathrm{d}V(x(t)) = \mathcal{L}V(x(t))\mathrm{d}t + \frac{\partial V}{\partial x} D(x(t))\mathrm{d}w(t).$$

Next, using (5.14) it follows that

$$\begin{aligned}
\int_0^{T_0} k(\|x_2(\tau)\|)\mathrm{d}\tau &= \int_0^{T_0} k(\|x_2(\tau)\|)\frac{r(V(x(\tau)))}{r(V(x(\tau)))}\mathrm{d}\tau \\
&\leq \int_0^{T_0} -\frac{\mathcal{L}V(x(\tau))}{r(V(x(\tau)))}\mathrm{d}\tau \\
&\leq \int_0^{T_0} -\frac{\mathrm{d}V(x(t))}{r(V(x(\tau)))} + \int_0^{T_0} \frac{1}{r(V(x(\tau)))}\frac{\partial V}{\partial x} D(x(\tau))\mathrm{d}w(\tau) \\
&= \int_0^{T_0} -\frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\mathrm{d}V(x(t)) + \int_0^{T_0} \frac{1}{r(V(x(\tau)))}\frac{\partial V}{\partial x} D(x(\tau))\mathrm{d}w(\tau). \qquad (5.21)
\end{aligned}$$

Once again, using Ito's (chain rule) formula it follows that

$$\begin{aligned}
\mathrm{d}\alpha(V(x(t))) &= \left[\frac{\partial \alpha(V(x))}{\partial x} f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)\frac{\partial^2 \alpha(V(x))}{\partial x^2} D(x)\right]\mathrm{d}t + \frac{\partial \alpha(V(x))}{\partial x}\mathrm{d}w(t) \\
&= \left[\frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\frac{\partial V(x)}{\partial x} f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)\frac{\partial}{\partial x}\left(\frac{\mathrm{d}\alpha(V)}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\right) D(x)\right]\mathrm{d}t \\
&\quad + \frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\frac{\partial V(x)}{\partial x}\mathrm{d}w(t) \\
&= \frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\left[\left(\frac{\partial V(x)}{\partial x} f(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)\frac{\partial^2 (V(x))}{\partial x^2} D(x)\right)\mathrm{d}t + \frac{\partial V(x)}{\partial x}\mathrm{d}w(t)\right] \\
&\quad + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right) D(x)\mathrm{d}t
\end{aligned}$$

$$= \frac{\mathrm{d}\alpha(V))}{\mathrm{d}V}\mathrm{d}V(x(t)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x)\mathrm{d}t. \tag{5.22}$$

Hence, it follows from (5.21) and (5.16) that

$$\begin{aligned}
\int_0^{T_0} k(\|x_2(\tau)\|)\mathrm{d}\tau &\le \int_0^{T_0} -\mathrm{d}\alpha(V(x(\tau))) + \int_0^{T_0}\frac{1}{r(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau)\\
&\quad + \int_0^{T_0}\frac{1}{2}\mathrm{tr}D^{\mathrm{T}}(x)\left(\frac{\partial V(x)}{\partial x}\right)^{\mathrm{T}}\frac{\mathrm{d}^2\alpha(V)}{\mathrm{d}V^2}\left(\frac{\partial V(x)}{\partial x}\right)D(x)\mathrm{d}\tau\\
&= \alpha(V(x(0))) - \alpha(V(x(T_0))) + \int_0^{T_0}\frac{1}{r(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau)\\
&\quad - \int_0^{T_0}\frac{r'(V)}{r^2(V)}\frac{1}{2}\mathrm{tr}\left(\frac{\partial V(x)}{\partial x}D^{\mathrm{T}}(x)\right)^{\mathrm{T}}\left(\frac{\partial V(x)}{\partial x}D(x)\right)\mathrm{d}\tau\\
&\le \int_0^{V(x(0))}\frac{\mathrm{d}v}{r(v)} - \int_0^{V(x(T_0))}\frac{\mathrm{d}v}{r(v)} + \int_0^{T_0}\frac{1}{r(V(x(\tau)))}\frac{\partial V}{\partial x}D(x(\tau))\mathrm{d}w(\tau).
\end{aligned} \tag{5.23}$$

Taking the expectation on both sides of (5.23) and using the fact that $x(0) \overset{\text{a.s.}}{=} x_0$ and $x(T_0) \overset{\text{a.s.}}{=} 0$ yields

$$\mathbb{E}^{x_0}\left[\int_0^{T_0} k(\|x_2(\tau)\|)\mathrm{d}\tau\right] \le \int_0^{V(x_0)}\frac{\mathrm{d}v}{r(v)}. \tag{5.24}$$

Next, since $q : [0,\infty) \to \mathbb{R}$ is continuously differentiable and satisfies (5.18), and, by assumption, the process $k(\|x_2(t))\|)$ is a positive $\mathcal{F}_t$-submartingale, it follows that $q(\cdot)$ is convex, monotonically increasing, and invertible. Hence, applying Jensen's inequality [41, p. 109], Fubini's theorem [4, p. 410], and the law of iterated expectation on the random variable $q(T(x_1(0), x_2(0)))$ yields

$$\begin{aligned}
\mathbb{E}^{x_0}\left[T(x_1(0), x_2(0))\right] &= q^{-1}(q(\mathbb{E}^{x_0}\left[T(x_1(0), x_2(0))\right]))\\
&\le q^{-1}(\mathbb{E}^{x_0}\left[q(T(x_1(0), x_2(0)))\right])\\
&= q^{-1}\left(\mathbb{E}^{x_0}\left[\int_0^{T_0}\mathbb{E}^{x_0}\left[k(\|x_2(\tau)\|)\right]\mathrm{d}\tau\right]\right)\\
&= q^{-1}\left(\mathbb{E}^{x_0}\left[\mathbb{E}^{x_0}\left[\int_0^{T_0} k(\|x_2(\tau)\|)\mathrm{d}\tau | T_0\right]\right]\right)\\
&= q^{-1}\left(\mathbb{E}^{x_0}\left[\int_0^{T_0} k(\|x_2(\tau)\|)\mathrm{d}\tau\right]\right)
\end{aligned}$$

$$\leq q^{-1}\left(\int_0^{V(x_0)}\frac{\mathrm{d}v}{r(v)}\right), \tag{5.25}$$

which shows that $T(x_1(0), x_2(0))$ is finite with probability one. Moreover, it follows from the stochastic finite-time stability of $\mathcal{G}$ with respect to x_1 and Proposition 5.1 that $T(\cdot,\cdot)$ can be extended to $\mathcal{H}_1^{\overline{\mathbb{R}}_+}$ and $T(0, x_{20}) \overset{\text{a.s.}}{=} 0$.

ii) Let $\rho > 0$, $x_{10} \in \mathbb{R}^{n_1}$, and $x_{20} \in \mathbb{R}^{n_2}$. Since $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions, it follows that, for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, \rho) > 0$ such that $\beta(\delta) \leq \alpha(\varepsilon)\rho$. Now, (5.14) implies that $V(x_1(t), x_2(t))$ is a (positive) supermartingale, and hence, it follows from (5.13) and (5.19) that, for all $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1}^{\mathcal{B}_\delta(0)} \times \mathcal{H}_{n_2}$,

$$\begin{aligned}\mathbb{P}^{x_0}\left(\sup_{t\geq 0}\|x_1(t)\| > \varepsilon\right) &\leq \sup_{t\geq 0}\frac{\mathbb{E}^{x_0}[\alpha(\|x_1(t)\|)]}{\alpha(\varepsilon)}\\ &\leq \sup_{t\geq 0}\frac{\mathbb{E}^{x_0}[V(x_1(t), x_2(t))]}{\alpha(\varepsilon)}\\ &\leq \frac{\mathbb{E}^{x_0}[V(x_1(0), x_2(0))]}{\alpha(\varepsilon)}\\ &\leq \frac{\beta(\delta)}{\alpha(\varepsilon)}\\ &\leq \rho.\end{aligned}$$

Hence, for every $x_1(0) \in \mathcal{H}_{n_1}^{\mathcal{B}_\delta(0)}$, $x_1(t) \in \mathcal{H}_{n_1}^{\mathcal{B}_\varepsilon(0)}$, $t \geq 0$, which proves uniform Lyapunov stability in probability with respect to x_1. Stochastic finite-time partial convergence follows as in the proof of i), implying global stochastic finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}. In addition, the existence of a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ that verifies (5.17) follows as in the proof of i).

iii) Global uniform stochastic finite-time stability of $\mathcal{G}$ with respect to x_1 directly follows from ii). Now, using similar arguments as in the proof of i), $q^{-1}(\cdot) = \frac{1}{k}(\cdot)$ directly follows from (5.18) with $k(\|x_2\|) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$. Now, the existence a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.20) holds follows as in the proof of i) and (5.17). Since $\mathbb{E}^{x_0}[T(x_1(0), x_2(0))])]$ is not a function of $x(t), t \geq 0$, strong stochastic finite-time convergence of $\mathcal{G}$ with respect to x_1 uniformly in x_{20} is immediate. Hence, the nonlinear stochastic dynamical system $\mathcal{G}$ is globally strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20}. □

Remark 5.2. If $r(V) = cV^{\theta}$, where $c > 0$ and $\theta \in (0,1)$, then $r(\cdot)$ satisfies (5.15) and (5.16). In this case, (5.17) becomes

$$\mathbb{E}^{x_0}[T(x_1(0), x_2(0)] \leq q^{-1}\left(\frac{V(x_{10}, x_{20})^{1-\theta}}{c(1-\theta)}\right).$$

For deterministic dynamical systems, this specialization recovers the finite-time, partial-state stability results given in [69] and the finite-time, full-state stability results given in [17].

Example 5.1. Consider the nonlinear stochastic dynamical system given by

$$\mathrm{d}x_1(t) = [-x_1(t) - x_1^{\frac{1}{3}}(t)x_2^2(t)]\mathrm{d}t + x_1(t)\mathrm{d}w(t), \qquad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \qquad t \geq 0, \tag{5.26}$$

$$\mathrm{d}x_2(t) = x_2(t)\mathrm{d}t + x_2(t)\mathrm{d}w(t), \qquad x_2(0) \overset{\text{a.s.}}{=} x_{20}. \tag{5.27}$$

To show that (5.26) and (5.27) is globally stochastic finite-time stable with respect to x_1 uniformly in x_{20}, consider the Lyapunov function candidate $V(x_1, x_2) = x_1^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (5.12), (5.13), and (5.19) hold, and

$$\mathcal{L}V(x_1, x_2) = \frac{4}{3}x_1^{\frac{1}{3}}\left(-x_1 - x_1^{\frac{1}{3}}x_2^2\right) + \frac{2}{9}x_1^{\frac{4}{3}} \leq -\frac{4}{3}x_2^2 x_1^{\frac{2}{3}} = -k(\|x_2\|)\left(V(x_1, x_2)\right)^{\frac{1}{2}}, \tag{5.28}$$

where $k(\|x_2\|) = \frac{4}{3}\|x_2\|^2$. Furthermore,

$$\mathcal{L}k(\|x_2\|) = \frac{8}{3}x_2^2 + \frac{4}{3}x_2^2 = 4x_2^2 \geq 0,$$

and hence, $k(\|x_2(t)\|)$ is a positive $\mathcal{F}_t$-submartingale. Hence, it follows from *iii*) of Theorem 5.1 that (5.26) and (5.27) is globally stochastic finite-time stable with respect to x_1 uniformly in x_{20}. △

The following results specialize Propositions 5.1 and Theorem 5.1 to nonlinear time-varying stochastic dynamical systems.

Proposition 5.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (5.5). Assume that $\mathcal{G}$ is globally stochastic finite-time stable and let $T : [0,\infty) \times \mathcal{H}_n^{\mathbb{R}^n\setminus\{0\}} \to \mathcal{H}_1^{[t_0,\infty)}$ be defined as in Definition 5.2. Then, for every $(t_0, x(t_0)) \in [0,\infty) \times \mathcal{H}_n$, there exists a solution $s(t, t_0, x(t_0))$, $t \geq t_0$, unique up to equivalence, to (5.5) such that $s(t, t_0, x(t_0)) \in \mathcal{H}_n$, $t \in [t_0, T(t_0, x(t_0)))$, and such that $\mathbb{P}^{x_0}(\sup_{t\geq T(t_0,x(t_0))} \|s(t, t_0, x(t_0))\| = 0) = 1$, where $T(t_0, 0) \triangleq 0$.

Proof: The result is a direct consequence of Proposition 5.1 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $D_1(x_1, x_2) = D_1(x_2, x_1) = D(t, x)$, $f_2(x_1, x_2) = 1$, $D_2(x_1, x_2) = D_2(x_2, x_1) = 0$, and $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$. □

Theorem 5.2. Consider the nonlinear dynamical system $\mathcal{G}$ given by (5.5). Then the following statements hold:

i) If there exist a two-times continuously differentiable function $V : [t_0,\infty) \times \mathbb{R}^n \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuous monotonically increasing function $k : [t_0,\infty) \to \mathbb{R}_+$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.15) and (5.16) hold, and, for all $(t, x) \in [t_0,\infty) \times \mathbb{R}^n$,

$$V(t, 0) = 0, \tag{5.29}$$

$$\alpha(\|x\|) \leq V(t, x), \tag{5.30}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t, x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t, x)\frac{\partial^2 V(t,x)}{\partial x^2} D(t, x) \leq -k(t)r(V(t, x)), \tag{5.31}$$

then $\mathcal{G}$ is globally stochastic finite-time stable. Moreover, there exists a a stochastic settling-time operator $T : [t_0,\infty) \times \mathcal{H}_n \to [t_0,\infty)$ such that

$$\mathbb{E}^{x_0}[T(t_0, x(t_0))] \leq q^{-1}\left(\int_0^{V(t_0,x_0)} \frac{\mathrm{d}v}{r(v)}\right), \qquad (t_0, x_0) \in [0,\infty) \times \mathbb{R}^n, \tag{5.32}$$

where $q : [0,\infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$\dot{q}(t) = k(t), \qquad q(0) = 0, \qquad t \geq 0. \tag{5.33}$$

ii) If there exists a two-times continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous monotonically increasing function $k : [t_0, \infty) \to \mathbb{R}_+$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.30)–(5.31) hold and

$$V(t, x) \leq \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{5.34}$$

then $\mathcal{G}$ is globally uniformly stochastic finite-time stable. Moreover, there exists a stochastic settling-time operator $T : [0, \infty) \times \mathcal{H}_n \to \mathcal{H}_1^{[t_0,\infty)}$ such that (5.32) holds.

iii) If there exists a two-times continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.15), (5.16), (5.30)–(5.31), and (5.34) hold with $k(t) = k \in \mathbb{R}_+$, $t \geq t_0$, then $\mathcal{G}$ is globally strongly uniformly stochastic finite-time stable. Moreover, there exists a a stochastic settling-time operator $T : [0, \infty) \times \mathcal{H}_n \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{t_0,x_0}[T(t_0, x(t_0))] \leq \int_0^{V(t_0,x_0)} \frac{\mathrm{d}v}{kr(v)}, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{5.35}$$

Proof: The result is a direct consequence of Theorem 5.1 using a similar construction as in the proof of Proposition 5.3. □

Example 5.2. Consider the nonlinear time-varying stochastic dynamical system given by

$$\mathrm{d}x(t) = [-x(t) - t\,(x(t))^{\frac{1}{3}}]\mathrm{d}t + x \sin t \mathrm{d}w(t), \qquad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq t_0. \tag{5.36}$$

To show that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (5.36) is globally uniformly stochastic finite-time stable, consider the Lyapunov function candidate $V(t, x) = x^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (5.29), (5.30), and (5.34) hold, and

$$\begin{aligned}\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x) + \frac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(t,x)\frac{\partial^2 V(t,x)}{\partial x^2} D(t,x) \\ = \frac{4}{3}x^{\frac{1}{3}}\left(-x - tx^{\frac{1}{3}}\right) + \frac{2}{9}x^{\frac{4}{3}}\sin^2 t \\ = -\frac{4}{3}tx^{\frac{2}{3}} - \frac{10}{9}x_1^{\frac{4}{3}} - \frac{2}{9}x_1^{\frac{4}{3}}\left(1 - \sin^2 t\right)\end{aligned}$$

$$\leq -\frac{4}{3}tx^{\frac{2}{3}}$$
$$= -k(t)\,(V(t,x))^{\frac{1}{2}}\,, \tag{5.37}$$

where $k(t) = \frac{4}{3}t > 0$, $t \geq t_0$. Hence, it follows from *iii*) of Theorem 5.2 that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ to (5.36) is globally uniformly stochastic finite-time stable. △

5.4. Stochastic Optimal Finite-Time, Partial-State Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquadratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear stochastic dynamical system given by (5.1) and (5.2). In particular, we prove stochastic finite-time partial stability of (5.1) and (5.2), and show that the nonlinear-nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t\right], \tag{5.38}$$

where $x_0 \triangleq [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (5.1) and (5.2), can be evaluated in a convenient form so long as (5.1) and (5.2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and is related to an underlying Lyapunov function satisfying an appropriate differential inequality.

Theorem 5.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (5.1) and (5.2) with performance measure (5.38). Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$, a continuous function $k : [0,\infty) \to \mathbb{R}_+$ such that $k(\|x_2(t)\|)$ is $\mathcal{F}_t$-submartingale such that (5.13)–(5.16) and (5.19) hold, and, for all $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$L(x_1, x_2) + V'(x_1, x_2)f(x_1, x_2) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) = 0. \tag{5.39}$$

Then the nonlinear stochastic dynamical system $\mathcal{G}$ is stochastic finite-time stable with respect to x_1 uniformly in x_{20} and there exists a settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.17) holds and satisfies (5.18). In addition, $J(x_{10}, x_{20}) = V(x_{10}, x_{20})$ for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Proof: Let $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (5.1) and (5.2). Then, it follows from Definition 4.1 and (5.14) that

$$\begin{aligned}\mathcal{L}V(x_1(t), x_2(t)) = V'(x_1(t), x_2(t)f(x_1(t), x_2(t)) + \frac{1}{2}\text{tr}\ D^{\mathrm{T}}(x_1(t), x_2(t)V''(x_1(t), x_2(t)) \\ \cdot D(x_1(t), x_2(t)) \leq -k(\|x_2(t)\|)r(V(x_1(t), x_2(t))), \qquad t \geq 0. \qquad (5.40)\end{aligned}$$

Thus, it follows from Theorem 5.1 that $\mathcal{G}$ is globally stochastic finite-time stable with respect to x_1 uniformly in x_{20}. In addition, it follows from Theorem 5.1 that there exists a settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.17) holds. Now, using Ito's (chain rule) formula it follows that the stochastic differential of $V(x_1(t), x_2(t))$ along the trajectories of $(x_1(t), x_2(t))$, $t \geq 0$, is given by

$$\mathrm{d}V(x_1(t), x_2(t)) = \mathcal{L}V(x_1(t), x_2(t))\mathrm{d}t + V'(x_1(t), x_2(t))D(x_1(t), x_2(t))\mathrm{d}w(t), \quad t \geq 0. \quad (5.41)$$

Hence, using (5.39) it follows that

$$\begin{aligned}&L(x_1(t), x_2(t))\mathrm{d}t + \mathrm{d}V(x_1(t), x_2(t)) \\ &\quad = \Bigg(L(x_1(t), x_2(t)) + \frac{\partial V(x_1(t), x_2(t))}{\partial x_1}f_1(x_1(t), x_2(t)) + \frac{\partial V(x_1(t), x_2(t))}{\partial x_2}f_2(x_1(t), x_2(t)) \\ &\qquad + \frac{1}{2}\text{tr}\ D_1^{\mathrm{T}}(x_1(t), x_2(t))\frac{\partial^2 V(x_1(t), x_2(t))}{\partial x_1^2}D_1(x_1(t), x_2(t)) \\ &\qquad + \frac{1}{2}\text{tr}\ D_2^{\mathrm{T}}(x_1(t), x_2(t))\frac{\partial^2 V(x_1(t), x_2(t))}{\partial x_2^2}D_2(x_1(t), x_2(t))\Bigg)\mathrm{d}t \\ &\qquad + \frac{\partial V(x(t))}{\partial x}D(x_1(t), x_2(t))\mathrm{d}w(t) \\ &\quad = \frac{\partial V(x(t))}{\partial x}D(x_1(t), x_2(t))\mathrm{d}w(t). \qquad (5.42)\end{aligned}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [t_0, \infty)$ be the first exit (stopping) time of the solution $x_1(t)$ and $x_2(t)$, $t \geq t_0$, from the set $\mathcal{B}_m(0) \times \mathbb{R}^{n_2}$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, integrating (5.42) over $[t_0, \min\{t_n, \tau_m\}]$, where $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\begin{aligned}\int_{t_0}^{\min\{t_n,\tau_m\}} & L(x_1(t), x_2(t))\ \mathrm{d}t \\ &= -\int_{t_0}^{\min\{t_n,\tau_m\}} \mathrm{d}V(x_1(t), x_2(t)) + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t) \\ &= V(x_1(t_0), x_2(t_0)) - V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\})) \\ &\quad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t). \end{aligned} \tag{5.43}$$

Next, taking the expectation on both sides of (5.43) yields

$$\begin{aligned}\mathbb{E}^{x_0} & \left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t))\ \mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\Big[V(x_1(t_0), x_2(t_0)) - V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\})) \\ &\quad + \int_{t_0}^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t)\Big] \\ &= V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\}))\right]. \end{aligned} \tag{5.44}$$

Now, noting that $L(x_1, x_2) \geq 0$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, the sequence of random variables $\{f_{n,m}\}_{n,m=0}^{\infty} \subseteq \mathcal{H}_1$, where

$$f_{n,m} \triangleq \int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t))\mathrm{d}t,$$

is a pointwise nondecreasing sequence in n and m of nonnegative $\mathcal{F}_t$-measurable random variables on Ω. Moreover, defining the improper integral

$$\int_{t_0}^{\infty} L(x_1(t), x_2(t))\mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from the Lebesgue monotone convergence theorem [3] that

$$\begin{aligned}\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0} & \left[\int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t))\ \mathrm{d}t\right] \\ &= \lim_{m\to\infty} \mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_{t_0}^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t))\ \mathrm{d}t\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_{t_0}^{\tau_m} L(x_1(t), x_2(t))\ \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x_1(t), x_2(t))\ \mathrm{d}t\right] \\
&= J(x_{10}, x_{20}).
\end{aligned} \tag{5.45}
$$

Next, since $\mathcal{G}$ is globally stochastic finite-time stable in probability with respect to x_1 uniformly in x_{20}, $V(\cdot,\cdot)$ is continuous, and $V(x_1(t), x_2(t))$, $t \geq t_0$, is positive supermartingale by (5.14) and [67, Lemma 5.4], it follows from [67, Theorem 5.1] that

$$
\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty} & \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right] \\
&= \lim_{m\to\infty}\mathbb{E}^{x_0}\left[\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right].
\end{aligned} \tag{5.46}
$$

Now, it follows from (5.13) and (5.19) that

$$
\begin{aligned}
V(x_{10}, x_{20}) &- \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty}\beta\left(\|x_1(\min\{t_n,\tau_m\})\|\right)\right] \\
&\leq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty} V(x_1(\min\{t_n,\tau_m\}), x_2(\min\{t_n,\tau_m\}))\right] \\
&\leq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty}\alpha\left(\|x_1(\min\{t_n,\tau_m\})\|\right)\right],
\end{aligned} \tag{5.47}
$$

and hence, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (5.44), using (5.45) and (5.46), and using the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$, we obtain

$$
\begin{aligned}
V(x_{10}, x_{20}) &- \mathbb{E}^{x_0}\left[\beta\left(\lim_{m\to\infty}\lim_{n\to\infty}\|x_1(\min\{t_n,\tau_m\})\|\right)\right] \\
&\leq J(x_{10}, x_{20}) \leq V(x_{10}, x_{20}) - \mathbb{E}^{x_0}\left[\alpha\left(\lim_{m\to\infty}\lim_{n\to\infty}\|x_1(\min\{t_n,\tau_m\})\|\right)\right].
\end{aligned} \tag{5.48}
$$

Now, $J(x_{10}, x_{20}) = V(x_{10}, x_{20})$ for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, is a direct consequence of (5.48) by using the fact that $\lim_{t\to T(x_1(0),x_2(0))} x_1(t) \overset{\text{a.s.}}{=} \lim_{t\to\infty} x_1(t) \overset{\text{a.s.}}{=} 0$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions. Finally, if $k(x_2) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$, then globally strong stochastic finite-time stability is direct consequence of *ii*) of Theorem 5.1. □

The following corollary to Theorem 5.3 considers the nonautonomous stochastic dynamical system (5.5) with performance measure

$$J(t_0, x_0) \triangleq \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(t, x(t))\mathrm{d}t\right], \tag{5.49}$$

where $L : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is jointly continuous in t and x, and $x(t)$, $t \geq t_0$, satisfies (5.5).

Corollary 5.1. Consider the nonlinear time-varying stochastic dynamical system (5.5) with performance measure (5.49). Assume that there exists a continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.15), (5.16), (5.30), (5.31), and (5.34) hold, and, for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$,

$$0 = \frac{\partial V(t,x)}{\partial t} + L(t,x) + \frac{\partial V(t,x)}{\partial x} f(t,x) + \frac{1}{2}\mathrm{tr}\; D^{\mathrm{T}}(t,x)\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x). \tag{5.50}$$

Then the nonlinear time-varying stochastic dynamical system (5.5) is globally uniformly stochastic finite-time stable and there exists a settling-time operator $T : [0, \infty) \times \mathcal{H}_n^{\mathcal{D}_0} \to \mathcal{H}_1^{[t_0,\infty)}$ such that (5.32) holds and satisfies (5.33). In addition, $J(t_0, x_0) = V(t_0, x_0)$ for all $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^n$.

Proof: The result is a direct consequence of Theorem 5.3. □

Next, we use the framework developed in Theorem 5.3 to obtain a characterization of stochastic optimal feedback controllers that guarantee closed-loop stochastic finite-time partial stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the stochastic Hamilton-Jacobi-Bellman equation. To address the problem of characterizing stochastic finite-time partially stabilizing feedback controllers, consider the controlled stochastic nonlinear dynamical system

$$\mathrm{d}x_1(t) = F_1(x_1(t), x_2(t), u(t))\mathrm{d}t + D_1(x_1(t), x_2(t), u(t))\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{5.51}$$

$$\mathrm{d}x_2(t) = F_2(x_1(t), x_2(t), u(t))\mathrm{d}t + D_2(x_1(t), x_2(t), u(t))\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{5.52}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{H}_{n_1}$, $F_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1}$, $F_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2}$, $D_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_1 \times d}$, $D_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2 \times d}$, and $F_1(0, x_2, 0) = 0$ and $D_1(0, x_2, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$.

Here once again we assume that $u(\cdot)$ satisfies sufficient regularity conditions such that (5.51) and (5.52) has a unique solution forward in time. Specifically, we assume that the control process $u(\cdot)$ in (5.51) and (5.52) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ such that $u(t) \in \mathcal{H}_m$, $t \geq 0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau), w(\tau), \tau \leq s$, and $x(0) = [x_1^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0)]^{\mathrm{T}}$, and hence, $u(\cdot)$ is nonanticipative. Furthermore, we assume $u(\cdot)$ takes values in a compact, metrizable set $\mathcal{U}$ and the uniform Lipschitz continuity (5.4) and growth restriction condition $\|f(x)\|^2 + \|D(x)\|_{\mathrm{F}}^2 \leq L^2(1 + \|x\|^2)$, $x \in \mathbb{R}^{(n_1+n_2)} \setminus \{0\}$, hold for the controlled drift and diffusion terms $F(x_1, x_2, u) \triangleq [F_1^{\mathrm{T}}(x_1, x_2, u), F_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$ and $D(x_1, x_2, u) \triangleq [D_1^{\mathrm{T}}(x_1, x_2, u), D_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$ uniformly in u. In this case, it follows from Theorem 2.2.4 of [5] that there exists a pathwise unique solution to (5.51) and (5.52) in $(\Omega, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, is called a *control law*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, where $\phi(\cdot, \cdot)$ is a control law and $x_1(t)$ and $x_2(t)$ satisfy (5.51) and (5.52), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot, \cdot)$ has values in $\mathcal{H}_m$.

Definition 5.3. Consider the controlled stochastic nonlinear dynamical system given by (5.51) and (5.52). The feedback control law $u = \phi(x_1, x_2)$ is *globally strongly stochastic finite-time stabilizing with respect to x_1 uniformly in x_{20}* if the closed-loop system (5.51) and (5.52) with $u = \phi(x_1, x_2)$ is globally strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20}.

Next, we present a main theorem for strong stochastic finite-time, partial-state stabilization characterizing feedback controllers that guarantee closed-loop stochastic finite-time partial stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, let $L : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m \to \mathbb{R}$ be jointly continuous in x_1, x_2, and u, and define the set of partial regulation controllers given by

$$\mathcal{S}(x_1(0), x_2(0)) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x_1(\cdot) \text{ given by (5.51)} \\ \text{satisfies } x_1(t) \overset{\text{a.s.}}{\to} 0 \text{ as } t \overset{\text{a.s.}}{\to} T(x_1(0), x_2(0))\},$$

where $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ is the stochastic settling-time operator. Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, that is, inputs corresponding to partial-state null convergent in probability solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost. In addition, since stochastic finite-time partial convergence is a stronger condition than asymptotic partial-state convergence in probability, $\mathcal{S}(x_1(0), x_2(0))$ includes the set of all partial-state null asymptotically convergent in probability controllers.

Theorem 5.4. Consider the controlled stochastic nonlinear dynamical system $\mathcal{G}$ given by (5.51) and (5.52) with performance functional

$$J(x_{10}, x_{20}, u(\cdot)) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty L(x_1(t), x_2(t), u(t))\,\mathrm{d}t\right], \tag{5.53}$$

where $u(\cdot)$ is an admissible control. Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$, and a control law $\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^m$ such that (5.13), (5.15), (5.16), and (5.19) hold, and

$$V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2, \phi(x_1, x_2))V''(x_1, x_2)D(x_1, x_2, \phi(x_1, x_2)) \\ \leq -r(V(x_1, x_2)), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{5.54}$$

$$\phi(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{5.55}$$

$$H(x_1, x_2, \phi(x)) = 0, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{5.56}$$

$$H(x_1, x_2, u) \geq 0, \qquad (x_1, x_2, u) \in \mathcal{D} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \tag{5.57}$$

$$H(x_1, x_2, u) \triangleq L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u) + \frac{1}{2}\text{tr } D^{\text{T}}(x_1, x_2, u)V''(x_1, x_2)D(x_1, x_2, u).$$

Then, with the feedback control $u = \phi(x_1, x_2)$, the closed-loop system given by (5.51) and (5.52) is globally strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20} and there exists a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.20) holds. In addition, if $(x_{10}, x_{20}) \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_2}$, then $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20})$ for all $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and the feedback control $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ minimizes $J(x_{10}, x_{20}, u(\cdot))$ in the sense that

$$J(x_{10}, x_{20}, \phi(\cdot, \cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_{10}, x_{20})} J(x_{10}, x_{20}, u(\cdot)). \tag{5.58}$$

Proof: Global strong stochastic finite-time stability with respect to x_1 uniformly in x_{20} are a direct consequence of (5.13), (5.19), and (5.54) by applying Theorem 5.1 to the closed-loop system given by (5.51) and (5.52) with $u = \phi(x_1, x_2)$. In addition, it follows from Theorem 5.1 that there exists a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.17) holds and $x_1(t) \stackrel{\text{a.s.}}{\to} 0$ as $t \stackrel{\text{a.s.}}{\to} T(x_1(0), x_2(0))$ for all initial conditions $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$. Furthermore, using (5.56), condition $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20})$ is a restatement of $J(x_{10}, x_{20}) = V(x_{10}, x_{20})$ as applied to the closed-loop system.

Next, let $(x_1(0), x_2(0)) \in \mathcal{H}_{n_1} \times \mathcal{H}_{n_2}$, let $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, and let $x_1(t)$ and $x_2(t)$, $t \geq 0$, be solutions of (5.51) and (5.52). Then, using Ito's (chain rule) formula it follows that the stochastic differential of $V(x_1(t), x_2(t)$ along the trajectories of $(x_1(t), x_2(t))$, $t \geq 0$, is given by

$$\mathrm{d}V(x_1(t), x_2(t)) = \mathcal{L}V(x_1(t), x_2(t))\mathrm{d}t + V'(x_1(t), x_2(t))D(x_1(t), x_2(t))\mathrm{d}w(t), \quad t \geq 0. \tag{5.59}$$

Hence, using the definition of $H(x_1, x_2, u)$ and (5.59) yields

$$\begin{aligned} L(x_1(t), x_2(t), u(t))\mathrm{d}t &= L(x_1(t), x_2(t), u(t))\mathrm{d}t - \mathrm{d}V(x_1(t), x_2(t)) + \mathcal{L}V(x_1(t), x_2(t))\mathrm{d}t \\ &\quad + V'(x_1(t), x_2(t))D(x_1(t), x_2(t))\mathrm{d}w(t) \\ &= -\mathrm{d}V(x_1(t), x_2(t)) + H(x_1(t), x_2(t), u(t))\mathrm{d}t \end{aligned}$$

$$+ V'(x_1(t), x_2(t))D(x_1(t), x_2(t))\mathrm{d}w(t), \qquad t \geq 0. \tag{5.60}$$

Now, it follows from (5.13) and (5.19) that

$$\begin{aligned} \mathbb{E}^{x_0}\left[\lim_{t\to T(x_1(0),x_2(0))} \alpha(\|x_1(t)\|)\right] &\leq \mathbb{E}^{x_0}\left[\lim_{t\to T(x_1(0),x_2(0))} V(x_1(t), x_2(t))\right] \\ &\leq \mathbb{E}^{x_0}\left[\lim_{t\to T(x_1(0),x_2(0))} \beta(\|x_1(t)\|)\right]. \end{aligned} \tag{5.61}$$

Using the continuity of $\alpha(\cdot)$ and $\beta(\cdot)$, and the fact that $\mathcal{G}$ is strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20}, it follows that

$$\begin{aligned} 0 = \mathbb{E}^{x_0}\left[\alpha\left(\lim_{t\to T(x_1(0),x_2(0))} \|x_1(t)\|\right)\right] &\leq \mathbb{E}^{x_0}\left[\lim_{t\to T(x_1(0),x_2(0))} V(x_1(t), x_2(t))\right] \\ &\leq \mathbb{E}^{x_0}\left[\beta\left(\lim_{t\to T(x_1(0),x_2(0))} \|x_1(t)\|\right)\right] = 0. \end{aligned} \tag{5.62}$$

Let $\{t_n\}_{n=0}^{\infty}$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [0, \infty)$ be the first exit (stopping) time of the solution $x_1(t)$ and $x_2(t)$, $t \geq 0$, from the set $\mathcal{B}_m(0) \times \mathbb{R}^{n_2}$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, integrating (5.60) over $[t_0, \min\{t_n, \tau_m\}]$, where $(n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\begin{aligned} \int_0^{\min\{t_n,\tau_m\}} & L(x_1(t), x_2(t), u(t))\,\mathrm{d}t \\ &= -\int_0^{\min\{t_n,\tau_m\}} \mathrm{d}V(x_1(t), x_2(t)) + \int_0^{\min\{t_n,\tau_m\}} H(x_1(t), x_2(t), u(t))\mathrm{d}t \\ &\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t) \\ &= V(x_1(0), x_2(0)) - V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\})) \\ &\quad + \int_0^{\min\{t_n,\tau_m\}} H(x_1(t), x_2(t), u(t))\mathrm{d}t \\ &\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t), x_2(t))\mathrm{d}w(t). \end{aligned} \tag{5.63}$$

Next, taking the expectation on both sides of (5.63) and using (5.57) yields

$$\mathbb{E}^{x_0}\left[\int_0^{\min\{t_n,\tau_m\}} L(x_1(t), x_2(t), u(t))\,\mathrm{d}t\right]$$

$$
\begin{aligned}
&= \mathbb{E}^{x_0}\Bigg[V(x_1(0),x_2(0)) - V(x_1(\min\{t_n,\tau_m\}),x_2(\min\{t_n,\tau_m\})) \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} H(x_1(t),x_2(t),u(t))\mathrm{d}t \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x_1(t),x_2(t))\mathrm{d}w(t)\Bigg] \\
&= V(x_{10},x_{20}) - \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n,\tau_m\}),x_2(\min\{t_n,\tau_m\}))\right] \\
&\quad + \mathbb{E}^{x_0}\left[\int_0^{\min\{t_n,\tau_m\}} H(x_1(t),x_2(t),u(t))\mathrm{d}t\right] \\
&\geq V(x_{10},x_{20}) - \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n,\tau_m\}),x_2(\min\{t_n,\tau_m\}))\right]. \qquad (5.64)
\end{aligned}
$$

Now, noting that for all $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$,

$$
\int_0^\infty |L(x_1(t),x_2(t),u(t))|\,\mathrm{d}t \overset{\text{a.s.}}{<} \infty,
$$

define the random variable

$$
g \triangleq \sup_{t\geq 0, m>0} \int_0^{\min\{t,\tau_m\}} |L(x_1(s),x_2(s),u(s))|\,\mathrm{d}s.
$$

In this case, the sequence of $\mathcal{F}_t$-measurable random variables $\{f_{n,m}\}_{n,m=0}^\infty \subseteq \mathcal{H}_1$ on Ω, where

$$
f_{n,m} \triangleq \int_0^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t),u(t))\mathrm{d}t,
$$

satisfies $|f_{n,m}| \overset{\text{a.s.}}{<} g$.

Next, defining the improper integral $\int_0^\infty L(x_1(t),x_2(t),u(t))\mathrm{d}t$ as the limit of a sequence of proper integrals, it follows from the dominated convergence theorem [3] that

$$
\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0}&\left[\int_0^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t),u(t))\mathrm{d}t\right] \\
&= \lim_{m\to\infty} \mathbb{E}^{x_0}\left[\lim_{n\to\infty}\int_0^{\min\{t_n,\tau_m\}} L(x_1(t),x_2(t),u(t))\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\int_0^{\tau_m} L(x_1(t),x_2(t),u(t))\mathrm{d}t\right] \\
&= \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(x_1(t),x_2(t),u(t))\mathrm{d}t\right] \\
&= J(x_{10},x_{20},u(\cdot)). \qquad (5.65)
\end{aligned}
$$

Finally, using the fact that $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$ and $V(\cdot, \cdot)$ is continuous, it follows that for every $m > 0$, $V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\}))$ is bounded for all $\{t_n\}_{n=0}^{\infty}$. Thus, using the dominated convergence theorem we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty} \mathbb{E}^{x_0}\left[V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\}))\right]$$
$$= \mathbb{E}^{x_0}\left[\lim_{m\to\infty}\lim_{n\to\infty} V(x_1(\min\{t_n, \tau_m\}), x_2(\min\{t_n, \tau_m\}))\right]. \qquad (5.66)$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (5.64) and using the fact $u(\cdot) \in \mathcal{S}(x_1(0), x_2(0))$, (5.62), (5.65), (5.66), and $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20})$ yields (5.58). □

Note that (5.56) is the steady-state, stochastic Hamilton-Jacobi-Bellman equation for the nonlinear controlled stochastic dynamical system (5.51) and (5.52) with performance criterion (5.53). Furthermore, conditions (5.56) and (5.57) guarantee optimality with respect to the set of admissible stochastic finite-time partially stabilizing controllers $\mathcal{S}(x_1(0), x_2(0))$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_1(0), x_2(0))$ is not required. In addition, the optimal strongly stochastic finite-time stabilizing with respect to x_1 uniformly in x_{20} *feedback* control law $u = \phi(x_1, x_2)$ is independent of the initial condition (x_{10}, x_{20}) and is given by

$$\phi(x_1, x_2) = \underset{u\in\mathcal{S}(x_1(0),x_2(0))}{\arg\min}\Big[L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u)$$
$$+ \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(x_1, x_2, u)V''(x_1, x_2)D(x_1, x_2, u)\Big]. \qquad (5.67)$$

Finally, we use Theorem 5.4 to provide a unification between optimal stochastic finite-time, partial-state stabilization and optimal stochastic finite-time control for stochastic nonlinear time-varying systems. Specifically, consider the controlled nonlinear time-varying stochastic dynamical system

$$\mathrm{d}x(t) = F(t, x(t), u(t))\mathrm{d}t + D(t, x(t), u(t))\mathrm{d}w(t), \qquad x(t_0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq t_0, \qquad (5.68)$$

with performance measure

$$J(t_0, x_0, u(\cdot)) \triangleq \mathbb{E}^{x_0}\left[\int_{t_0}^{\infty} L(t, x(t), u(t))\mathrm{d}t\right], \tag{5.69}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n$, $u(t) \in \mathcal{H}_m$, and $L : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $F : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and $D : [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times d}$ are jointly continuous in t, x, and u. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0, x(t_0)) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (5.68)}$$
$$\text{satisfies } x(t) \overset{\text{a.s.}}{\to} 0 \text{ as } t \overset{\text{a.s.}}{\to} T(t_0, x(t_0))\},$$

where $T : [t_0, \infty) \times \mathcal{H}_n \to \mathcal{H}_1^{(t_0,\infty)}$ is the stochastic settling-time operator.

Corollary 5.2. Consider the controlled stochastic nonlinear time-varying dynamical system (5.68) with performance measure (5.69) where $u(\cdot)$ is an admissible control. Assume that there exists a two-times continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$, and a control law $\phi : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m$ such that (5.15), (5.16), (5.30), (5.31), and (5.34) hold, and

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}F(t,x,\phi(t,x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,\phi(t,x))\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x,\phi(t,x))$$
$$\leq -r(V(t,x), \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{5.70}$$

$$\phi(t, 0) = 0, \quad t \in [t_0, \infty), \tag{5.71}$$

$$L(t,x,\phi(t,x)) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}F(t,x,\phi(t,x)) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,\phi(t,x))$$
$$\cdot\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x,\phi(t,x)) = 0, \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{5.72}$$

$$L(t,x,u) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}F(t,x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u)\frac{\partial^2 V(t,x)}{\partial x^2}$$
$$\cdot D(t,x,u) \geq 0, \quad (t,x,u) \in [t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m. \tag{5.73}$$

Then, with the feedback control $u = \phi(t, x)$, the closed-loop system given by (5.68) is globally strongly uniformly stochastic finite-time stable and there exists a stochastic settling-time operator $T : [t_0, \infty) \times \mathcal{H}_n \to \mathcal{H}_1^{[t_0,\infty)}$ such that (5.35) holds. In addition, if $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^n$, then $J(t_0, x_0, \phi(\cdot,\cdot)) = V(t_0, x_0)$ for all $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^n$, and the feedback control $u(\cdot) = \phi(\cdot, x(\cdot))$ minimizes $J(t_0, x_0, u(\cdot))$ in the sense that

$$J(t_0, x_0, \phi(\cdot,\cdot)) = \min_{u(\cdot)\in\mathcal{S}(t_0,x(t_0))} J(t_0, x_0, u(\cdot)). \tag{5.74}$$

Proof: The proof is a direct consequence of Theorem 5.4. □

Note that (5.72) and (5.73) give the classical stochastic Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u\in\mathcal{S}(t_0,x(t_0))} \left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x} F(t,x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u) \frac{\partial^2 V(t,x)}{\partial x^2} \cdot D(t,x,u) \right], \quad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \tag{5.75}$$

which characterizes the optimal control

$$\phi(t,x) = \arg\min_{u\in\mathcal{S}(t_0,x(t_0))} \left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x} F(t,x,u) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u) \frac{\partial^2 V(t,x)}{\partial x^2} D(t,x,u) \right] \tag{5.76}$$

for time-varying stochastic systems on a finite or infinite interval.

5.5. Finite-Time Stabilization for Affine Dynamical Systems and Connections to Inverse Optimal Control

In this section, we specialize the results of Section 5.4 to stochastic nonlinear affine dynamical systems of the form

$$\mathrm{d}x_1(t) = \left[f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t))u(t) \right] \mathrm{d}t + D_1(x_1(t), x_2(t))\mathrm{d}w(t), \tag{5.77}$$

$$\mathrm{d}x_2(t) = \left[f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t))u(t) \right] \mathrm{d}t + D_2(x_1(t), x_2(t))\mathrm{d}w(t), \tag{5.78}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{H}_{n_1}$, $x_2(t) \in \mathcal{H}_{n_2}$, $u(t) \in \mathcal{H}_m$, and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $G_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times m}$, $G_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times m}$, $D_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times d}$, and $D_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times d}$ are such that $f_1(0, x_2) = 0$ and $D_1(0, x_2) = 0$ for all

$x_2 \in \mathbb{R}^{n_2}$, and $f_1(\cdot,\cdot)$, $f_2(\cdot,\cdot)$, $G_1(\cdot,\cdot)$, $G_2(\cdot,\cdot)$, $D_1(\cdot,\cdot)$, and $D_2(\cdot,\cdot)$ are jointly continuous in x_1 and x_2 in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Furthermore, we consider performance integrands $L(x_1,x_2,u)$ of the form

$$L(x_1,x_2,u) = L_1(x_1,x_2) + L_2(x_1,x_2)u + u^{\mathrm{T}}R_2(x_1,x_2)u, \qquad (x_1,x_2,u)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\times\mathbb{R}^{m}, \tag{5.79}$$

where $L_1 : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2} \to \mathbb{R}$, $L_2 : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}^{1\times m}$, and $R_2(x_1,x_2) \geq N(x_1) > 0$, $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, so that (5.53) becomes

$$J(x_{10},x_{20},u(\cdot)) = \mathbb{E}^{x_0}\bigg[\int_0^\infty \big[L_1(x_1(t),x_2(t)) + L_2(x_1(t),x_2(t))u(t) + u^{\mathrm{T}}(t)R_2(x_1(t),x_2(t))u(t)\big]\mathrm{d}t\bigg]. \tag{5.80}$$

For the statement of the next result, define

$$f(x_1,x_2) \triangleq [f_1^{\mathrm{T}}(x_1,x_2), f_2^{\mathrm{T}}(x_1,x_2)]^{\mathrm{T}}, \qquad G(x_1,x_2) \triangleq [G_1^{\mathrm{T}}(x_1,x_2), G_2^{\mathrm{T}}(x_1,x_2)]^{\mathrm{T}}, \tag{5.81}$$

$$F(x_1,x_2,u) \triangleq f(x_1,x_2) + G(x_1,x_2)u, \qquad D(x_1,x_2) \triangleq [D_1^{\mathrm{T}}(x_1,x_2), D_2^{\mathrm{T}}(x_1,x_2)]^{\mathrm{T}}. \tag{5.82}$$

Theorem 5.5. Consider the controlled stochastic nonlinear affine dynamical system (5.77) and (5.78) with performance measure (5.80). Assume that there exists a two-times continuously differentiable function $V : \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to\mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.15) and (5.16) hold, and for all $(x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$,

$$\alpha(\|x_1\|) \leq V(x_1,x_2) \leq \beta(\|x_1\|), \tag{5.83}$$

$$V'(x_1,x_2)\bigg[f(x_1,x_2) - \frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2)L_2^{\mathrm{T}}(x_1,x_2) - \frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2) \cdot G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\bigg] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2) \leq -r(V(x_1,x_2)), \tag{5.84}$$

$$L_2(0,x_2) = 0, \tag{5.85}$$

$$0 = L_1(x_1,x_2) + V'(x_1,x_2)f(x_1,x_2) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2) - \frac{1}{4}\Big[V'(x_1,x_2)$$

$$\cdot G(x_1, x_2) + L_2(x_1, x_2)\Big] R_2^{-1}(x_1, x_2) \Big[V'(x_1, x_2)G(x_1, x_2) + L_2(x_1, x_2)\Big]^{\mathrm{T}}. \qquad (5.86)$$

Then, with the feedback control

$$u = \phi(x_1, x_2) = -\frac{1}{2} R_2^{-1}(x_1, x_2)\big[L_2(x_1, x_2) + V'(x_1, x_2)G(x_1, x_2)\big]^{\mathrm{T}}, \qquad (5.87)$$

the closed-loop system (5.77) and (5.78) is globally strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20} and there exists a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.20) holds. In addition, $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)) = V(x_{10}, x_{20})$, $(x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and the performance measure (5.80) is minimized in the sense of (5.58).

Proof: The result is a consequence of Theorem 5.4 with

$$L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}} R_2(x_1, x_2)u.$$

Specifically, the feedback control law (5.87) follows from (5.67) by setting

$$\frac{\partial}{\partial u}\Big[L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}} R_2(x_1, x_2)u + V'(x_1, x_2)\big[f(x_1, x_2) + G(x_1, x_2)u\big] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2)\Big] = 0. \qquad (5.88)$$

Now, with $u = \phi(x_1, x_2)$ given by (5.87), conditions (5.83), (5.84), (5.15), (5.16), and (5.86) imply (5.13), (5.19), (5.54), and (5.56), respectively.

Next, since $V(\cdot, \cdot)$ is continuously differentiable and, by (5.83), $V(0, x_2)$, $x_2 \in \mathbb{R}^{n_2}$, is a local minimum of $V(\cdot, \cdot)$, it follows that $V'(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and hence, it follows from (5.85) and (5.87) that $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, which implies (5.55). Finally, since

$$\begin{aligned} L(x_1, x_2, u) + V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)u] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) \\ = L(x_1, x_2, u) + V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)u] \\ + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) - L(x_1, x_2, \phi(x_1, x_2)) - V'(x_1, x_2) \\ \cdot [f(x_1, x_2) + G(x_1, x_2)\phi(x_1, x_2)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1, x_2)V''(x_1, x_2)D(x_1, x_2) \end{aligned}$$

$$= [u - \phi(x_1, x_2)]^{\mathrm{T}} R_2(x_1, x_2)[u - \phi(x_1, x_2)]$$

$$\geq 0, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{5.89}$$

condition (5.57) holds. The result now follows as a direct consequence of Theorem 5.4. □

The following corollary to Theorem 5.5 considers the nonautonomous dynamical system

$$\mathrm{d}x(t) = [f(t, x(t)) + G(t, x(t))u(t)]\mathrm{d}t + D(t, x(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{5.90}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) = \mathbb{E}^{t_0, x_0}\left[\int_{t_0}^{\infty} \left[L_1(t, x(t)) + L_2(t, x(t))u(t) + u^{\mathrm{T}}(t) R_2(t, x(t)) u(t)\right] \mathrm{d}t\right], \tag{5.91}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n$ and $u(t) \in \mathcal{H}_m$, $f : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $G : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$, and $D : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are such that $f(t, 0) = 0$, $D(t, 0) = 0$ for all $t \in [t_0, \infty)$, $f(\cdot, \cdot)$ and $G(\cdot, \cdot)$ are jointly continuous in x_1 and x_2 on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $L_1 : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, $L_2 : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^{1 \times m}$, and $R_2(t, x) \geq N(x) > 0$, $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$.

Corollary 5.3. Consider the controlled stochastic nonlinear affine dynamical system (5.90) with performance measure (5.91). Assume that there exists a two-times continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.15) and (5.16) hold, and, for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$,

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \tag{5.92}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x}\left[f(t, x) - \frac{1}{2} G(t, x) R_2^{-1}(t, x) L_2^{\mathrm{T}}(t, x) - \frac{1}{2} G(t, x) R_2^{-1}(t, x)\right.$$
$$\left. \cdot G^{\mathrm{T}}(t, x)\left(\frac{\partial V(t, x)}{\partial x}\right)^{\mathrm{T}}\right] + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(t, x) \frac{\partial^2 V(t, x)}{\partial x^2} D(t, x) \leq -r(V(t, x)), \tag{5.93}$$

$$L_2(t, 0) = 0, \tag{5.94}$$

$$0 = L_1(t, x) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x) + \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(t, x) \frac{\partial^2 V(t, x)}{\partial x^2} D(t, x)$$
$$- \frac{1}{4}\left[\frac{\partial V(t, x)}{\partial x} G(t, x) + L_2(t, x)\right] R_2^{-1}(t, x) \left[\frac{\partial V(t, x)}{\partial x} G(t, x) + L_2(t, x)\right]^{\mathrm{T}}. \tag{5.95}$$

Then, with the feedback control

$$u = \phi(t,x) = -\frac{1}{2}R_2^{-1}(t,x)\left[L_2(t,x) + \frac{\partial V(t,x)}{\partial x}G(t,x)\right]^{\mathrm{T}}, \tag{5.96}$$

the closed-loop system (5.90) is globally strongly uniformly stochastic finite-time stable and there exists a stochastic settling-time operator $T : [0,\infty) \times \mathcal{H}_n \to \mathcal{H}_1^{[t_0,\infty)}$ such that (5.35) holds. In addition, $J(t_0, x_0, \phi(\cdot, x(\cdot))) = V(t_0, x_0)$ for all $(t_0, x_0) \in [0,\infty) \times \mathbb{R}^n$, and the performance measure (5.91) is minimized in the sense of (5.74).

Proof: The proof is a direct consequence of Theorem 5.5. □

Finally, we construct state feedback controllers for stochastic nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [34,42].

Theorem 5.6. Consider the controlled nonlinear affine stochastic dynamical system (5.77) and (5.78) with performance measure (5.80). Assume there exists a two-times continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.83)–(5.85) hold. Then, with the feedback control (5.87), the closed-loop system given by (5.77) and (5.78) is globally strongly stochastic finite-time stable with respect to x_1 uniformly in x_{20} and there exists a stochastic settling-time operator $T : \mathcal{H}_{n_1} \times \mathcal{H}_{n_2} \to \mathcal{H}_1^{[0,\infty)}$ such that (5.20) holds. In addition, the performance functional (5.80), with

$$\begin{aligned} L_1(x_1,x_2) =& \phi^{\mathrm{T}}(x_1,x_2)R_2(x_1,x_2)\phi(x_1,x_2) - V'(x_1,x_2)f(x_1,x_2) \\ &- \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2), \end{aligned}$$

is minimized in the sense of (5.58) and $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20})$.

Proof: The proof is identical to the proof of Theorem 5.5. □

The following corollary to Theorem 5.3 considers the nonautonomous stochastic dynamical system (5.90) with performance measure (5.91).

Corollary 5.4. Consider the controlled stochastic nonlinear affine dynamical system (5.90) with performance measure (5.91). Assume there exists a two-times continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a continuously differentiable function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that (5.92)–(5.94) holds. Then, with the feedback control (5.96), the closed-loop system given by (5.90) is globally strongly uniformly stochastic finite-time stable and there exists a stochastic settling-time operator $T : [0, \infty) \times \mathcal{H}_n \to \mathcal{H}_1^{[t_0,\infty)}$ such that (5.35) holds. In addition, the performance functional (5.91), with

$$L_1(t,x) = \phi^{\mathrm{T}}(t,x)R_2(t,x)\phi(t,x) - \frac{\partial V(t,x)}{\partial t} - \frac{\partial V(t,x)}{\partial x}f(t,x) - \frac{1}{2}\mathrm{tr}\, D^{\mathrm{T}}(t,x)\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x),$$

is minimized in the sense of (5.74) and $J(t_0, x_0, \phi(\cdot, x(\cdot))) = V(t_0, x_0)$.

Proof: The proof is identical to the proof of Theorem 5.6. □

5.6. Illustrative Numerical Examples

In this section, we provide two numerical examples to highlight the stochastic finite-time, partial-state stabilization framework developed in the chapter.

Example 5.3. Consider the controlled nonlinear time-varying stochastic dynamical system given by

$$\mathrm{d}x_1(t) = \left[-x_1^{\frac{1}{3}}(t) + tx_2(t) + u_1(t)\right]\mathrm{d}t + x_1(t)\sin x_2(t)\mathrm{d}w(t), \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{5.97}$$

$$\mathrm{d}x_2(t) = \left[-x_2^{\frac{1}{3}}(t) - tx_1(t) + u_2(t)\right]\mathrm{d}t + \sin t x_2(t)\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}. \tag{5.98}$$

Note that (5.97) and (5.98) can be cast in the form of (5.90) with $f(t,x) = [-x_1^{\frac{1}{3}} + tx_2,\ -x_2^{\frac{1}{3}} - tx_1]^{\mathrm{T}}$, $G(t,x) = I_2$, and $D(t,x) = [x_1 \sin x_2,\ \sin t x_2]^{\mathrm{T}}$, where $x \triangleq [x_1\ x_2]^{\mathrm{T}}$. For this example, we use Corollary 5.4 to construct an inverse optimal globally strongly uniformly finite-time stabilizing control law for (5.97) and (5.98). Let $V(t,x) = V(x) = \frac{1}{2}\alpha x_1^2 + \frac{1}{2}\alpha x_2^2$, $\alpha > 0$, let $L(t,x,u) = L_1(t,x) + L_2(t,x)u + u^{\mathrm{T}}R_2(t,x)u$, where $R_2(t,x) > 2\alpha I_2$, and note that

$L_2(t,x) = \alpha x^{\mathrm{T}}$ satisfies (5.94) so that the inverse optimal control law (5.96) is given by $\phi(t,x) = -\alpha R_2^{-1}(t,x)x$. In this case, the performance functional (5.91), with $L_1(t,x) = \alpha^2 x^{\mathrm{T}} R_2^{-1} x + \alpha x_1^{\frac{4}{3}} + \alpha x_2^{\frac{4}{3}} - \frac{1}{2}\alpha x_1^2 \sin^2 x_2 - \frac{1}{2}\alpha x_2^2 \sin^2 t$, is minimized in the sense of (5.74).

Next, since (5.92) holds with $\alpha(\|x\|) = \beta(\|x\|) = V(x)$ and

$$\begin{aligned}
&\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}\Bigg[f(t,x) - \frac{1}{2}G(t,x)R_2^{-1}(t,x)L_2^{\mathrm{T}}(t,x) - \frac{1}{2}G(t,x)R_2^{-1}(t,x)\\
&\quad\cdot G^{\mathrm{T}}(t,x)\left(\frac{\partial V(t,x)}{\partial x}\right)^{\mathrm{T}}\Bigg] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x)\frac{\partial^2 V(t,x)}{\partial x^2}D(t,x)\\
&\qquad = -\alpha x_1^{\frac{4}{3}} - \alpha x_2^{\frac{4}{3}} - \alpha^2 x^{\mathrm{T}} R_2^{-1} x + \tfrac{1}{2}\alpha x_1^2 \sin^2 x_2 + \tfrac{1}{2}\alpha x_2^2 \sin^2 t\\
&\qquad \leq -\alpha(x_1^{\frac{4}{3}} + x_2^{\frac{4}{3}}) - \alpha x^{\mathrm{T}}(\alpha R_2^{-1} - \tfrac{1}{2}I_2)x\\
&\qquad \leq -\rho(V(t,x))^{\frac{2}{3}}, \qquad (t,x) \in [0,\infty) \times \mathbb{R}^2,
\end{aligned} \tag{5.99}$$

where $\rho = 2^{\frac{2}{3}}\alpha^{\frac{1}{3}}$, (5.93) holds with $r(v) = \rho v^{\frac{2}{3}}$. Hence, it follows from Corollary 5.4 that the feedback control law $\phi(t,x)$ is globally strongly uniformly stochastic finite-time stabilizing. Moreover, there exists a stochastic settling-time operator $T : [0,\infty) \times \mathcal{H}_2 \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{t_0,x_0}[T(t_0,x(t_0))] \leq \frac{3}{2}\|x_0\|^{\frac{2}{3}}, \qquad (t_0,x_0) \in [0,\infty) \times \mathbb{R}^2, \tag{5.100}$$

and

$$J(t_0,x_0,\phi(t,x(\cdot))) = \frac{1}{2}\alpha\|x_0\|^2, \qquad (t_0,x_0) \in [0,\infty) \times \mathbb{R}^2. \tag{5.101}$$

For $x(0) \overset{\text{a.s.}}{=} [2,\ -1]^{\mathrm{T}}$, $\alpha = 1$, and $R_2 = 4I_2$, Figure 6.2 shows a sample state trajectory of the controlled system versus time, whereas Figure 6.3 shows the corresponding sample control signal versus time. △

Example 5.4. Consider the spacecraft with one axis of symmetry and with stochastic disturbances given by

$$\mathrm{d}\omega_1(t) = [I_{23}\omega_2(t)\omega_3(t) + u_1(t)]\mathrm{d}t + \sigma\omega_1(t)\mathrm{d}w(t), \quad \omega_1(0) \overset{\text{a.s.}}{=} \omega_{10}, \quad t \geq 0, \tag{5.102}$$

$$\mathrm{d}\omega_2(t) = [-I_{23}\omega_3(t)\omega_1(t) + u_2(t)]\mathrm{d}t + \sigma\omega_2(t)\mathrm{d}w(t), \qquad \omega_2(0) \overset{\text{a.s.}}{=} \omega_{20}, \tag{5.103}$$

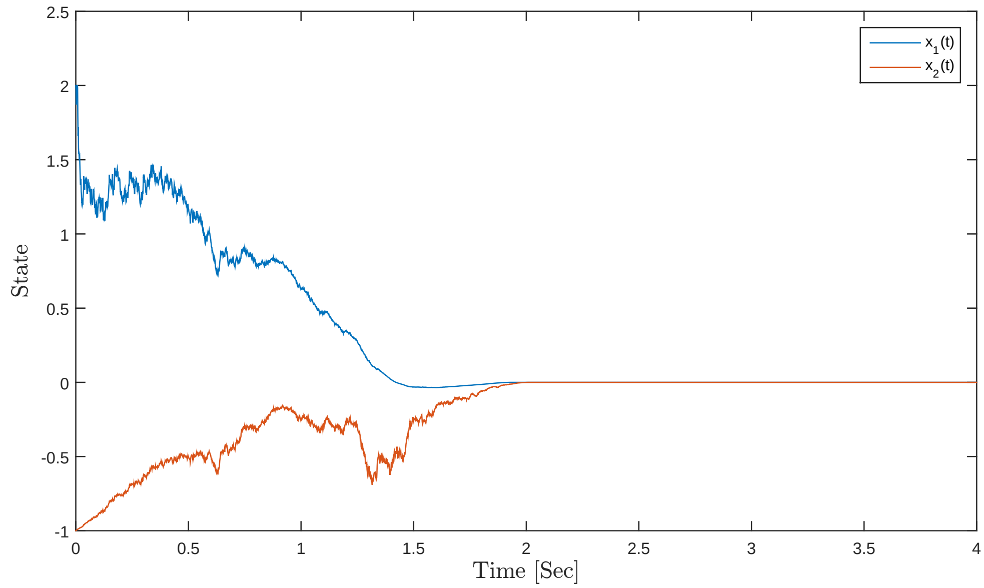

Figure 5.1: Closed-loop system sample trajectory versus time.

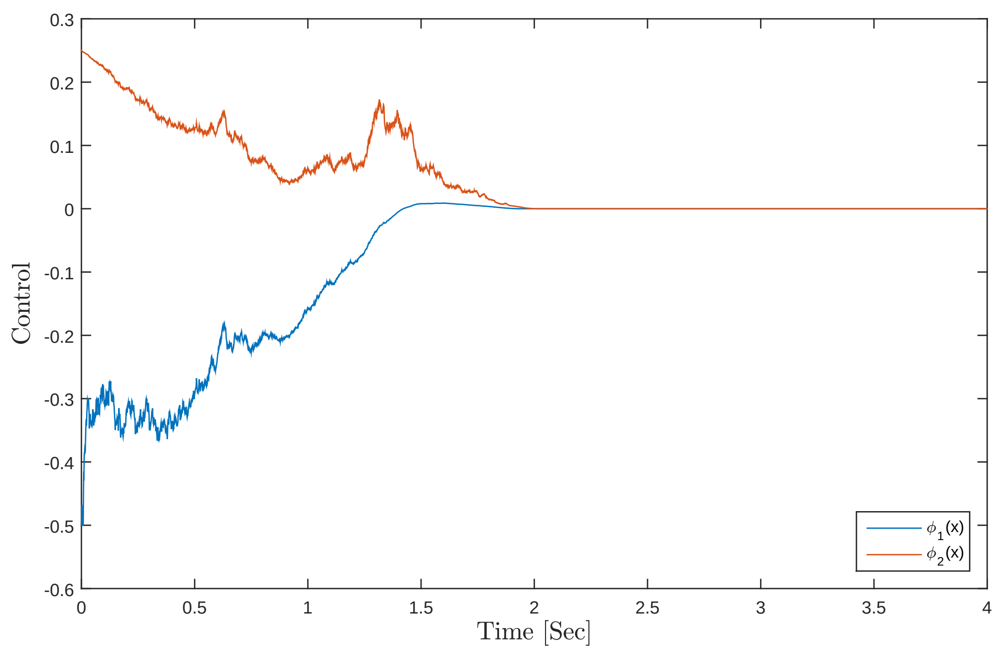

Figure 5.2: Control signal versus time.

$$\mathrm{d}\omega_3(t) = [\alpha_3 u_1(t) + \alpha_4 u_2(t)]\mathrm{d}t + \sigma\omega_3(t)\mathrm{d}w(t), \qquad \omega_3(0) \overset{\text{a.s.}}{=} \omega_{30}, \tag{5.104}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\omega_1 : [0, \infty) \to \mathcal{H}_1$, $\omega_2 : [0, \infty) \to \mathcal{H}_1$, and $\omega_3 : [0, \infty) \to \mathcal{H}_1$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, $w(\cdot)$ is a standard Wiener process with intensity $\sigma > 0$, α_3 and $\alpha_4 \in \mathbb{R}$, and u_1 and u_2 are the spacecraft control moments. Here, the state-dependent disturbances can be used to capture perturbations in atmospheric drag for low altitude (i.e., $<$ 600 km) satellites from the Earth's residual atmosphere as well as J_2 perturbations due to the nonspherical mass distribution of the Earth and its nonuniform mass density. For details see [33, 62].

For this example, we apply Theorem 5.6 to find an *inverse optimal* globally partial-state stabilizing control law $u = [u_1,\ u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1,\ \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the spacecraft is stochastic finite-time spin-stabilized about its third principle axis of inertia, that is, the stochastic dynamical system (5.102)–(5.104) is globally stochastic strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$. Note that (5.102)–(5.104) can be cast in the form of (5.77) and (5.78), with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = \left[I_{23}\omega_2\omega_3,\ -I_{23}\omega_3\omega_1,\ 0\right]^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix} 1 & 0 & \alpha_3 \\ 0 & 1 & \alpha_4 \end{bmatrix}^{\mathrm{T}}$, and $D(x_1, x_2) = \sigma\,[x_1, x_2]^{\mathrm{T}}$.

To construct an inverse optimal controller for (5.102) and (5.103), let $V(x_1, x_2) = \left(x_1^{\mathrm{T}} x_1\right)^{\alpha}$, where $\frac{1}{2} < \alpha < 1$, $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}u$, and let

$$L_2(x_1, x_2) = 2\left[I_{23}\omega_3\omega_2 + \sigma^2(2\alpha - 1)\omega_1,\quad -I_{23}\omega_3\omega_1 + \sigma^2(2\alpha - 1)\omega_2\right]. \tag{5.105}$$

Now, the inverse optimal control law (5.87) is given by

$$u = \phi(x_1, x_2) = \begin{bmatrix} -\alpha\omega_1\|x_1\|^{2(\alpha-1)} - I_{23}\omega_3\omega_2 - \sigma^2(2\alpha - 1)\omega_1 \\ -\alpha\omega_2\|x_1\|^{2(\alpha-1)} + I_{23}\omega_3\omega_1 - \sigma^2(2\alpha - 1)\omega_2 \end{bmatrix} \tag{5.106}$$

and the performance functional (5.80), with

$$L_1(x_1, x_2) = \left(\alpha\|x_1\|^{2\alpha-1} + \sigma^2(2\alpha - 1)\|x_1\|\right)^2 + (I_{23}\omega_3\|x_1\|)^2 - \alpha\sigma^2(2\alpha - 1)\|x_1\|^{\alpha}, \tag{5.107}$$

is minimized in the sense of (5.58). Furthermore, since (5.83) holds with $\alpha(\|x_1\|) = \beta(\|x_1\|) = V(x_1, x_2)$ and, since, for all $(x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}$,

$$V'(x_1, x_2)\left[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)L_2^{\mathrm{T}}(x_1,x_2) - \frac{1}{2}G(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\right]$$

$$+\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x_1,x_2)V''(x_1,x_2)D(x_1,x_2) = -2\alpha^2(x^{\mathrm{T}}x)^{2\alpha-1} = -2\alpha^2(V(x_1,x_2))^{\frac{2\alpha-1}{\alpha}}, \quad (5.108)$$

(5.84) holds with $r(v) = 2\alpha^2 v^{\frac{2\alpha-1}{\alpha}}$. Hence, with the feedback control law $\phi(x_1,x_2)$ given by (5.106), the closed-loop system (5.102) and (5.103) is globally stochastic finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a stochastic settling-time operator $T : \mathcal{H}_2 \times \mathcal{H}_1 \to \mathcal{H}_1^{[0,\infty)}$ such that

$$\mathbb{E}^{x_0}[T(x_1(0),x_2(0)] \le \frac{1}{2\alpha(1-\alpha)}\left(\omega_{10}^2+\omega_{20}^2\right)^{2(1-\alpha)}, \qquad (x_{10},x_{20}) \in \mathbb{R}^2\times\mathbb{R}, \qquad (5.109)$$

where $x_{10} = [\omega_{10},\ \omega_{20}]^{\mathrm{T}}$ and $x_{20} = \omega_{30}$, and

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))) = \left(\omega_{10}^2+\omega_{20}^2\right)^{2\alpha}, \qquad (x_{10},x_{20}) \in \mathbb{R}^2\times\mathbb{R}. \qquad (5.110)$$

For $I_1 = I_2 = 4\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = -2\,\mathrm{Hz}$, $\omega_{20} = 2\,\mathrm{Hz}$, $\omega_{30} = 1\,\mathrm{Hz}$, $\alpha_3 = \frac{\sqrt{2}}{2}$, $\alpha_4 = -\frac{\sqrt{2}}{2}$, $\sigma = \frac{1}{3}$, and $\alpha = 4/5$, Figure 5.3 shows a sample trajectory along with the standard deviation of the state trajectories $x_1(t)$, $t \ge 0$, of the controlled system versus time. Figure 5.4 shows a sample path along with the standard deviation of the corresponding control signal versus time for $x_1(0) \overset{\text{a.s.}}{=} [-2,\ 2]^{\mathrm{T}}$ and $x_2(0) \overset{\text{a.s.}}{=} 1$ for 15 sample paths. Finally, $J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))) = 5.28\,\mathrm{Hz}^2$. △

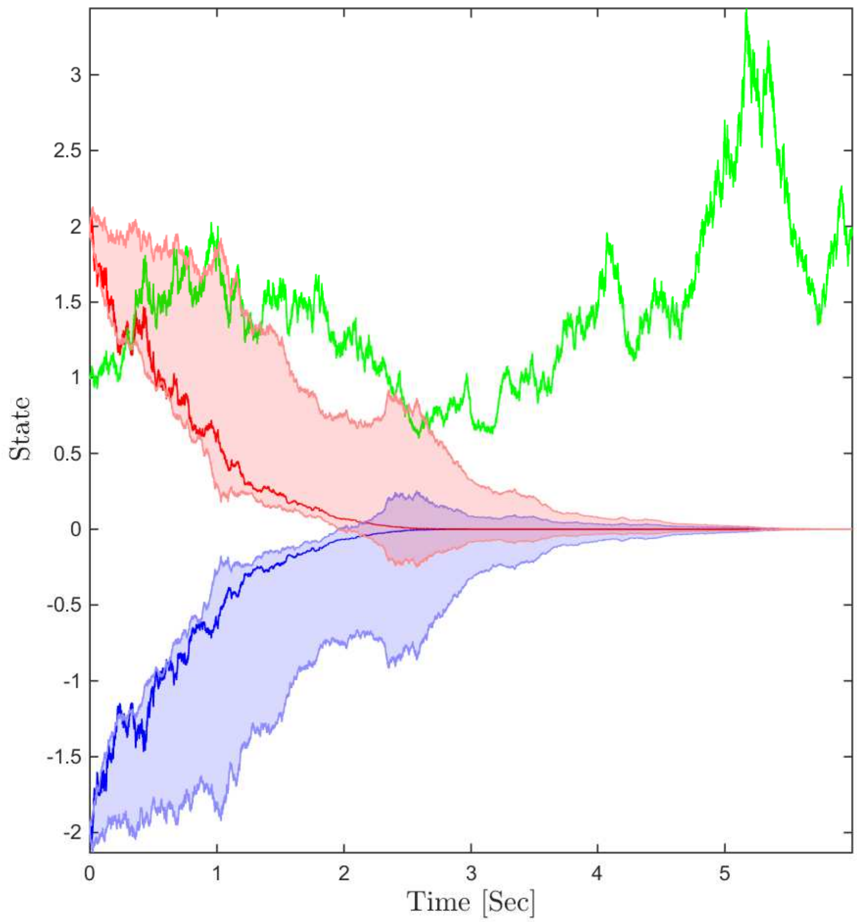

Figure 5.3: A sample trajectory along with the sample standard deviation of the closed-loop system trajectories versus time; $\omega_1(t)$ in blue, $\omega_2(t)$ in red, and in green is a sample trajectory of $\omega_3(t)$.

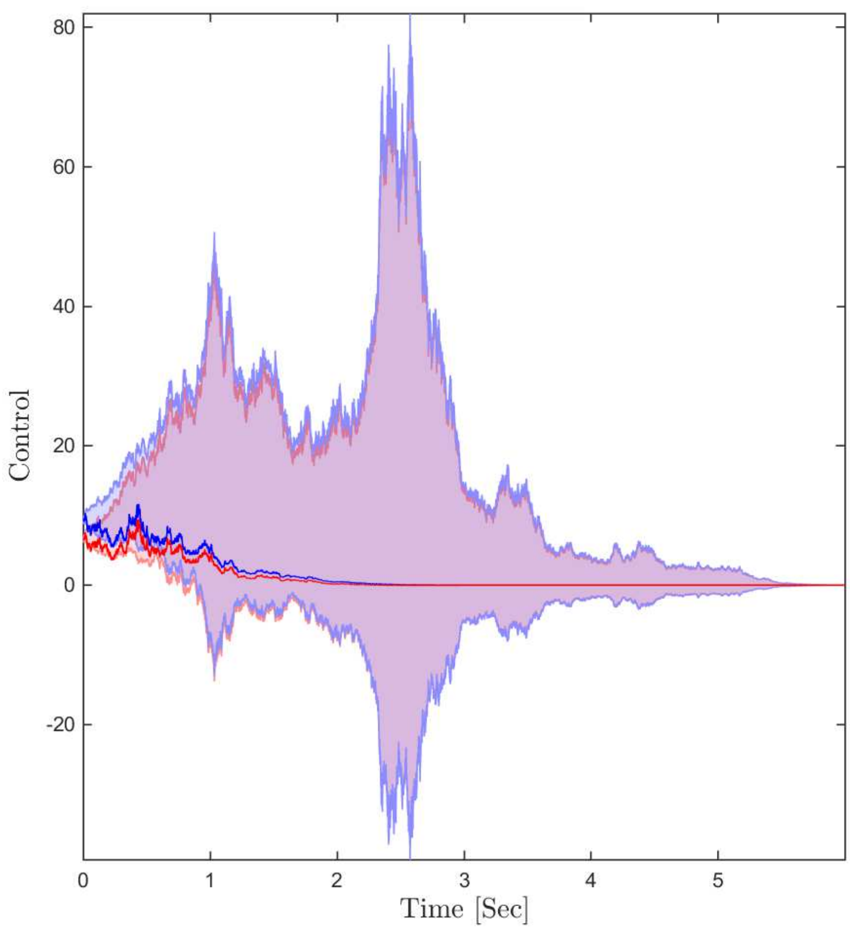

Figure 5.4: A sample path along with the sample standard deviation of the control signal versus time; $u_1(t)$ in blue and $u_2(t)$ in red.

Chapter 6

Stochastic Differential Games and Inverse Optimal Control and Stopper Policies

6.1. Introduction

Building on the results of Chapter 3, in this chapter we present a two-player stochastic differential game framework for designing optimal feedback control and stopper policies for each player. Specifically, we consider feedback stochastic optimal control policies for attaining higher utilities or lower costs over an infinite horizon involving a nonlinear-nonquadratic performance functional. The performance functional can be evaluated in closed form as long as the nonlinear-nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability in probability of the nonlinear differential game problem. This Lyapunov function is shown to be the solution of the steady-state stochastic Hamilton-Jacobi-Isaacs equation. The overall framework provides the foundation for extending linear-quadratic controller and stopper policies for stochastic differential games to nonlinear-nonquadratic differential games with polynomial and multilinear cost functionals.

More specifically, in Section 6.2, we state a nonlinear-nonquadratic stochastic differential game problem and provide sufficient conditions for characterizing optimal nonlinear feedback controller and stopper policies guaranteeing asymptotic stability in probability of the closed-loop system and providing a minimax solution to the differential game problem. Then, in

Section 6.3, we develop an inverse optimal framework tailored to the stochastic differential game problem. This result is then used to derive optimal nonlinear feedback controller and stopper policies that minimize and maximize general polynomial and multilinear performance criteria. Finally, in Section 6.4, we provide two illustrative examples that highlight the proposed stochastic differential game framework.

6.2. Stochastic Differential Games and Optimal Control and Stopper Policies

In this section, we consider a two-player stochastic differential game problem, wherein the two players (i.e., controller strategy and stopper strategy) attempt to control the state of the system so as to minimize and maximize, respectively, a given nonlinear-nonquadratic performance measure. Our framework considers control and stopper strategies involving a notion of optimality that is directly related to a specified Lyapunov function. Specifically, sufficient conditions for optimal game strategies are given in a form that corresponds to a steady-state version of the stochastic Hamilton-Jacobi-Isaacs equation.

To address the problem of characterizing stochastic optimal stabilizing feedback laws for the controller and stopper of the stochastic differential game, consider the two-player stochastic differential game described by the nonlinear stochastic differential game problem $\mathcal{G}$ given by

$$\mathrm{d}x(t) = F(x(t), u(t), v(t))\mathrm{d}t + D(x(t), u(t), v(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.1}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $x(0) \in \mathcal{H}_n^{x_0}$, $\mathcal{D}$ is an open with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_{m_1}^{U_1}$, $U_1 \subseteq \mathbb{R}^{m_1}$ is open with $0 \in U_1$, $v(t) \in \mathcal{H}_{m_2}^{U_2}$, $U_2 \subseteq \mathbb{R}^{m_2}$ is open with $0 \in U_2$, $w(\cdot)$ is a d-dimensional independent standard Wiener process, $x(0)$ is independent of $(w(t) - w(0)), t \geq 0$, $F : \mathcal{D} \times U_1 \times U_2 \to \mathbb{R}^n$ is jointly continuous in x, u, and v with $F(0,0,0) = 0$, and $D : \mathcal{D} \times U_1 \times U_2 \to \mathbb{R}^{n \times d}$ is jointly continuous in x, u, and v with $D(0,0,0) = 0$.

Here we assume that $u(\cdot)$ and $v(\cdot)$ satisfy sufficient regularity conditions such that the system (6.1) has a unique solution forward in time. Specifically, we assume that the control and stopper policies in (6.1) are restricted to the class of *admissible* policies consisting of measurable functions $u(\cdot)$ and $v(\cdot)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ such that $u(t) \in \mathcal{H}_m^{U_1}$, $t \geq 0$, $v(t) \in \mathcal{H}_{m_2}^{U_2}$, $t \geq 0$, and, for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau)$, $v(\tau)$, $\tau \leq s$, and $u(\cdot)$ and $v(\cdot)$ take values in compact metrizable sets $\mathcal{U}_1$ and $\mathcal{U}_2$. Furthermore, we assume that the uniform Lipschitz continuity and growth conditions (2.4) and (2.5) hold for the controlled drift and diffusion terms $F(x, u, v)$ and $D(x, u, v)$ uniformly in u and v. In this case, it follows from Theorem 2.2.4 of [5] that there exists a pathwise unique solution to (6.1) in $(\Omega, \{\mathcal{F}_{t\geq 0}\}, \mathbb{P}^{x_0})$.

A measurable function $\phi : \mathcal{D} \to U_1$ (resp., $\psi : \mathcal{D} \to U_2$) is called a *control* (resp., *stopper*) *law.* If $u(t) = \phi(x(t))$ and $v(t) = \psi(x(t))$, $t \geq 0$, where $\phi(\cdot)$ and $\psi(\cdot)$ are control and stopper laws, and $x(t)$, $t \geq 0$, satisfies (6.1), then we call $u(\cdot)$ and $v(\cdot)$ *feedback control* and *feedback stopper laws.* Note that the feedback control (resp., stopper) law is an admissible control (resp., stopper) since $\phi(\cdot)$ (resp., $\psi(\cdot)$) has values in U_1 (resp., U_2). Given a control and a stopper law $\phi(\cdot)$ and $\psi(\cdot)$, and feedback control and stopper laws $u(t) = \phi(x(t))$ and $v(t) = \psi(x(t))$, $t \geq 0$, the *closed-loop system* (6.1) has the form

$$\mathrm{d}x(t) = F(x(t), \phi(x(t)), \psi(x(t)))\mathrm{d}t + D(x(t), \phi(x(t)), \psi(x(t)))\mathrm{d}w(t) \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0. \tag{6.2}$$

Next, we define the strategies along with the strategy spaces for the controller and stopper. For the statement of this result, let $L : \mathcal{D} \times U_1 \times U_2 \to \mathbb{R}$ be jointly continuous in x, u, and v, and let $\mathbb{1}_{[0,\tau_m]}(t)$ denote the indicator function defined on the set $[0, \tau_m]$, $m \in \mathbb{Z}_+$, that is,

$$\mathbb{1}_{[0,\tau_m]}(t) \triangleq \begin{cases} 1, & \text{if } t \in [0, \tau_m], \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, let $\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}$ denote the set of all sample trajectories of (6.1) with controls law $u(\cdot)$ and stopper law $v(\cdot)$ for which $\lim_{t\to\infty} \|x(t,\omega)\| = 0$ and $x(\{t \geq 0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}$,

$\omega \in \Omega$. Finally, define

$$\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \triangleq \begin{cases} 1, & \text{if } x(\{t \geq 0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}, \\ 0, & \text{otherwise.} \end{cases}$$

Here we consider the nonlinear-nonquadratic performance measure (or payoff function) given by

$$J(x_0, u(\cdot), v(\cdot)) \triangleq \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_0^\infty L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t\right], \qquad (6.3)$$

and, we assume that the regulation controller policy and strategy is such that the payoff function (6.3) is well defined. This is shown in Theorem 6.1 below. Furthermore, since there can exist several different types of solutions for a differential game problem, including minimax, Nash, and Stackelberg solutions [8], the following definitions are required for developing our minimax differential game framework.

Definition 6.1 [39, Def. 2.1]. An *Elliott-Kalton strategy* for the *stopper* is a mapping $\beta : \mathcal{U}_1 \to \mathcal{U}_2$ such that if $u(t) \overset{\text{a.s.}}{\equiv} \hat{u}(t)$, then $v(t) = \beta(\hat{u}(t))$ maximizes (6.3) with a *stopper strategy space* $\mathcal{S}_{\text{EK}}$ consisting of the set of all such stopper strategies. Similarly, the *Elliott-Kalton strategy* for the *controller* is a mapping $\alpha : \mathcal{U}_2 \to \mathcal{U}_1$ such that if $v(t) \overset{\text{a.s.}}{\equiv} \hat{v}(t)$, then $u(t) = \alpha(\hat{v}(t))$ minimizes (6.3) with a *controller strategy space* $\mathcal{C}_{\text{EK}}$ consisting of the set of all such controller strategies.

Given $x_0 \in \mathbb{R}^n$, the *upper* and *lower values* of the stochastic differential game are defined by

$$V^+(x_0) \triangleq \sup_{\mathcal{S}_{\text{EK}}} \inf_{\mathcal{U}_1} J(x_0, u(\cdot), \beta(u(\cdot))), \qquad (6.4)$$

$$V^-(x_0) \triangleq \inf_{\mathcal{C}_{\text{EK}}} \sup_{\mathcal{U}_2} J(x_0, \alpha(v(\cdot)), v(\cdot)). \qquad (6.5)$$

Note that in general $V^-(x_0) \leq V^+(x_0)$; however, if the *Isaacs minimax condition* holds [40], then $V^+(x_0) = V^-(x_0)$. Moreover, it follows from the Definition 6.1 that if (6.5) holds, then the controller has an advantage over the stopper. Specifically, in this case, the controller has an informational advantage at each time t, and hence, the class of admissible controller strategies $\mathcal{C}_{\mathrm{EK}}$ should be restricted to eliminate this advantage. This is done by restricting the set of controller strategies $\mathcal{C}_{\mathrm{EK}}$ to a set of progressively measurable strategies.

Definition 6.2. A controller strategy $\alpha(\cdot) \in \mathcal{C}_{\mathrm{EK}}$ is *strictly progressively measurable* if, for each stopper strategy $\beta(\cdot) \in \mathcal{S}_{\mathrm{EK}}$, the equations $u = \alpha(v)$ and $v = \beta(u)$ have a unique solution. In this case, we denote the set of strictly progressively measurable strategies by $\mathcal{C}_{\mathrm{S}}$.

If the controller strategy $\alpha(\cdot) \in \mathcal{C}_{\mathrm{S}}$, then it can be shown that ([39])

$$V^+(x_0) \leq \inf_{\mathcal{C}_{\mathrm{S}}} \sup_{\mathcal{U}_2} J(x_0, \alpha(v(\cdot)), v(\cdot)), \tag{6.6}$$

and hence, $V^+(x_0) = V^-(x_0)$.

Definition 6.3. The *saddle point property* for the upper game holds if there exists a real valued function $V : \mathcal{D} \to \mathbb{R}$ and controller and stopper strategies $\alpha_\varepsilon(\cdot) \in \mathcal{C}_{\mathrm{S}}$ and $\beta_\varepsilon(\cdot) \in \mathcal{S}_{\mathrm{EK}}$, where $\varepsilon > 0$, such that the following conditions hold:

i) $V(x) - \varepsilon \leq \inf_{u \in \mathcal{U}_1} J(x, u(\cdot), \beta_\varepsilon(u(\cdot)))$.

ii) $\sup_{v \in \mathcal{U}_2} J(x, \alpha_\varepsilon(v(\cdot)), v(\cdot)) \leq V(x) + \varepsilon$.

Next, we present a main theorem for the two-player stochastic differential game problem characterizing feedback controller and stopper policies that guarantee closed-loop stability in probability and optimize a nonlinear-nonquadratic performance functional. For the statement of this result, define the set of stochastic regulation controller policies and strategies given by

$$\mathcal{S}(x_0, \rho) \triangleq \Big\{ u(\cdot) \in \mathcal{U}_1 : u(\cdot) = \alpha(v(\cdot)), v(\cdot) \in \mathcal{U}_2, \alpha(\cdot) \in \mathcal{C}_{\mathrm{S}}, \text{ and } x(\cdot) \text{ given by (6.1)}$$
$$\text{is such that } \mathbb{P}^{x_0}\Big(\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}\Big) \geq 1 - \rho \Big\}.$$

Theorem 6.1. Consider the nonlinear two-player stochastic differential game problem (6.1) with performance functional (6.3) where the stopper strategy is an Elliott-Kalton strategy and the controller strategy is a strictly progressively measurable strategy. Assume that there exist a two-times continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and control and stopper laws $\phi : \mathcal{D} \to U_1$ and $\psi : \mathcal{D} \to U_2$, with strategies $\alpha(\cdot) \in \mathcal{C}_{\rm S}$ and $\beta(\cdot) \in \mathcal{S}_{\rm EK}$, such that

$$V(0) = 0, \tag{6.7}$$

$$V(x) > 0, \qquad x \in \mathcal{D}, \qquad x \neq 0, \tag{6.8}$$

$$\phi(0) = 0, \tag{6.9}$$

$$\psi(0) = 0, \tag{6.10}$$

$$V'(x)F(x,\phi(x),\psi(x)) + \frac{1}{2}{\rm tr}\ D^{\rm T}(x,\phi(x),\psi(x))V''(x)D(x,\phi(x),\psi(x)) < 0, \quad x \neq 0, \tag{6.11}$$

$$H(x,\phi(x),\psi(x)) = 0, \qquad x \in \mathcal{D}, \tag{6.12}$$

$$H(x,u,\psi(x)) \geq 0, \qquad x \in \mathcal{D}, \qquad u \in U_1, \tag{6.13}$$

$$H(x,\phi(x),v)) \leq 0, \qquad x \in \mathcal{D}, \qquad v \in U_2, \tag{6.14}$$

$$\phi(x) = \alpha(\psi(x)), \tag{6.15}$$

$$\psi(x) = \beta(\phi(x)), \tag{6.16}$$

where

$$H(x,u,v) \triangleq L(x,u,v) + V'(x)F(x,u,v) + \frac{1}{2}{\rm tr}\ D^{\rm T}(x,u,v)V''(x)D(x,u,v). \tag{6.17}$$

Then, with the feedback control and stopper policies $u(\cdot) = \phi(x(\cdot))$ and $v(\cdot) = \psi(x(\cdot))$, the zero solution $x(t) \overset{\rm a.s.}{\equiv} 0$ of the closed-loop system (6.2) is locally asymptotically stable in probability and, for every $\rho \in (0,1)$, there exist $\delta = \delta(\rho)$ and $\mathfrak{B}_{x_0}^{\phi(x(\cdot)),\psi(x(\cdot))}$ with $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot)),\psi(x(\cdot))}\right) \geq 1-\rho$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$,

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot))) = V(x_0). \tag{6.18}$$

In addition, if $x_0 \in \mathcal{B}_\delta(0)$, then the feedback control and stopper policies $u(\cdot) = \phi(x(\cdot))$ and

$v(\cdot) = \psi(x(\cdot))$ optimize $J(x_0, u(\cdot), v(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0,\rho)} \max_{v(\cdot)\in\mathcal{U}_2} J(x_0, u(\cdot), v(\cdot)). \tag{6.19}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U_1 = \mathbb{R}^{m_1}$, $U_2 = \mathbb{R}^{m_2}$, and

$$V(x) \to \infty \text{ as } \|x\| \to \infty, \tag{6.20}$$

then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (6.2) is globally asymptotically stable in probability and (6.19) holds with $\rho = 0$ and $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot)),\psi(x(\cdot))}\right) = 1$, $x_0 \in \mathbb{R}^n$.

Proof. Local and global asymptotic stability in probability are a direct consequence of (6.7)–(6.11) by applying Theorem 3.1 to the closed-loop system (6.2). Consequently, for every $\rho \in (0,1)$, there exist $\delta = \delta(\rho)$ and a set of sample trajectories $x(\{t \geq 0\}, \omega) \in \mathfrak{B}_{x_0}^{\phi(x(\cdot)),\psi(x(\cdot))}$ such that, for all $x_0 \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}$, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot)),\psi(x(\cdot))}\right) \geq 1 - \rho$.

Next, let $x_0 \in \mathcal{B}_\delta(0)$, let $u(\cdot) \in \mathcal{S}(x_0, \rho)$, and let $x(t)$, $t \geq 0$, be the solution of (6.1). Then, using Itô's chain rule formula it follows that

$$\begin{aligned} L(x(t), u(t), v(t)\mathrm{d}t + \mathrm{d}V(x(t)) = &\left(L(x,u,v) + V'(x)F(x,u,v) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,u,v)V''(x)\right. \\ &\left.\cdot D(x,u,v)\right)\mathrm{d}t + \frac{\partial V(x)}{\partial x}D(x,u,v)\mathrm{d}w(t), \end{aligned}$$

and hence,

$$\begin{aligned} L(x(t), u(t), v(t))\mathrm{d}t = &- \ \mathrm{d}V(x(t)) + H(x(t), u(t), v(t))\mathrm{d}t \\ &+ \frac{\partial V(x(t))}{\partial x}D(x(t), u(t), v(t))\mathrm{d}w(t). \end{aligned} \tag{6.21}$$

Let $\{t_n\}_{n=0}^\infty$ be a monotonic sequence of positive numbers with $t_n \to \infty$ as $n \to \infty$, $\tau_m : \Omega \to [0, \infty)$ be the first exit (stopping) time of the solution $x(t)$, $t \geq 0$, from the set $\mathcal{B}_m(0)$, and let $\tau \triangleq \lim_{m\to\infty} \tau_m$. Now, multiplying (6.21) with $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)$ and integrating over $[0, \min\{t_n, \tau_m\}]$, where $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, yields

$$\int_0^{\min\{t_n,\tau_m\}} L(x(t), u(t), v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\ \mathrm{d}t$$

$$
\begin{aligned}
&= -\int_0^{\min\{t_n,\tau_m\}} \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}V(x(t)) + \int_0^{\min\{t_n,\tau_m\}} H(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}t \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} \frac{\partial V(x(t))}{\partial x} D(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}w(t) \\
&= V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \\
&\quad + \int_0^{\min\{t_n,\tau_m\}} H(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}t \\
&\quad + \int_0^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t),u(t),v(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}w(t).
\end{aligned}
\tag{6.22}
$$

Next, taking the expectation on both sides of (6.22) yields

$$
\begin{aligned}
&\mathbb{E}^{x_0}\left[\int_0^{\min\{t_n,\tau_m\}} L(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\ \mathrm{d}t\right] \\
&\qquad = \mathbb{E}^{x_0}\Bigg[V(x(t_0))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) - V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \\
&\qquad\quad + \int_0^{\min\{t_n,\tau_m\}} H(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}t \\
&\qquad\quad + \int_0^{t_n} \frac{\partial V(x(t))}{\partial x} D(x(t),u(t),v(t))\mathbb{1}_{[t_0,\tau_m]}(t)\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}w(t)\Bigg] \\
&\qquad = V(x_0)\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}\right) - \mathbb{E}^{x_0}\left[V(x(\min\{t_n,\tau_m\}))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right] \\
&\qquad\quad + \mathbb{E}^{x_0}\left[\int_0^{\min\{t_n,\tau_m\}} H(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}t\right].
\end{aligned}
\tag{6.23}
$$

Next, let $\mathfrak{B}_{x_0}^m$ denote the set of all the sample trajectories of $x(t)$, $t \geq 0$, such that $\tau_m = \infty$ and note that, by regularity of solutions [67, p. 75], $\mathbb{P}^{x_0}(\mathfrak{B}_{x_0}^m) \to 1$ as $m \to \infty$. Now, noting that for all $u(\cdot) \in \mathcal{S}(x_0,\rho)$,

$$
\int_0^\infty \left|L(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right| \mathrm{d}t \overset{\text{a.s.}}{<} \infty,
$$

let the random variable

$$
g \triangleq \sup_{t\geq 0, m>0} \int_0^{\min\{t,\tau_m\}} \left|L(x(s),u(s),v(s))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right| \mathrm{d}s.
$$

In this case, the sequence in n and m of $\mathcal{F}_t$-measurable random variables $\{f_{m,n}\}_{m,n=0}^\infty \subseteq \mathcal{H}_1$ on Ω for all $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$, where

$$
f_{m,n} \triangleq \int_0^{\min\{t_n,\tau_m\}} L(x(t),u(t),v(t))\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\mathrm{d}t
$$

satisfies $|f_{m,n}| \overset{\text{a.s.}}{<} g$, $(n,m) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. Now, defining the improper integral

$$\int_0^\infty L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t$$

as the limit of a sequence of proper integrals, it follows from the dominated convergence theorem [3] that

$$\begin{aligned}
\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}^{x_0} &\left[\int_0^{\min\{t_n,\tau_m\}} L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t\right] \\
&= \lim_{m\to\infty} \mathbb{E}^{x_0} \left[\lim_{n\to\infty} \int_0^{\min\{t_n,\tau_m\}} L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0} \left[\lim_{m\to\infty} \int_0^{\tau_m} L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t\right] \\
&= \mathbb{E}^{x_0} \left[\int_0^{\infty} L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t\right] \\
&= J\left(x_0, u(\cdot), v(\cdot)\right) \mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}\right). \qquad (6.24)
\end{aligned}$$

Next, using the fact that $u(\cdot) \in \mathcal{S}(x_0, \rho)$ and $V(\cdot)$ is continuous, it follows that for every $m > 0$, $V(x((\min\{t_n, \tau_m\}))$ is bounded for all $n \in \mathbb{Z}_+$. Thus, using the dominated convergence theorem [3] and the fact that $\|x(t,\omega)\| \to 0$ as $t \to \infty$ for all $x(\{t \geq 0\}, \omega) \in \mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}$, we obtain

$$\begin{aligned}
\lim_{m\to\infty} \lim_{n\to\infty} \mathbb{E}^{x_0} &\left[V(x(\min\{t_n, \tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right] \\
&= \lim_{m\to\infty} \mathbb{E}^{x_0} \left[\lim_{n\to\infty} V(x(\min\{t_n, \tau_m\})) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right] \\
&= \mathbb{E}^{x_0} \left[\lim_{m\to\infty} V(x(\tau_m)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right] \\
&= \mathbb{E}^{x_0} \left[V\left(\lim_{m\to\infty} x(\tau_m)\right) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega)\right] \\
&= 0. \qquad (6.25)
\end{aligned}$$

Now, taking the limit as $n \to \infty$ and $m \to \infty$ on both sides of (6.23) and using (6.24), (6.25), and (6.12), yields (6.18). Since $\alpha(\cdot) \in \mathcal{C}_{\mathrm{S}}$, there exist $\phi(\cdot)$ and $\psi(\cdot)$ such that (6.15) and (6.16) are satisfied and it follows from Theorem 3.4 of [39] that the saddle point property holds. Moreover, using the Issacs minimax condition it follows from (6.12)–(6.16) that $u(\cdot) = \phi(x(\cdot))$ and $v(\cdot) = \psi(x(\cdot))$ are the optimal feedback control and stopper laws and (6.19) holds.

Finally, for $\mathcal{D} = \mathbb{R}^n$ global asymptotic stability in probability of closed-loop system is direct consequence of the radially unbounded condition on $V(\cdot)$, and hence, $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(x(\cdot)),\psi(x(\cdot))}\right)$ $= 1$ for all $x_0 \in \mathbb{R}^n$ and (6.19) holds for $\rho = 0$. □

Note that (6.12) is the steady-state stochastic Hamilton-Jacobi-Isaacs equation. To see this, recall that the lower and upper stochastic Hamilton-Jacobi-Isaacs equations have the form ([24])

$$\frac{\partial}{\partial t}V^-(t,x(t)) + H^-\left(t, x(t), \frac{\partial}{\partial x}V^-(t,x(t)), \frac{\partial^2}{\partial x^2}V^-(t,x(t))\right) = 0, \quad t \geq 0, \tag{6.26}$$

$$\frac{\partial}{\partial t}V^+(t,x(t)) + H^+\left(t, x(t), \frac{\partial}{\partial x}V^+(t,x(t)), \frac{\partial^2}{\partial x^2}V^+(t,x(t))\right) = 0, \tag{6.27}$$

with lower and upper Hamiltonians given by

$$H^-(t,x(t),p,X) = \sup_{u\in U_1} \inf_{v \in U_2} H(t,x,p,X,u,v), \tag{6.28}$$

$$H^+(t,x(t),p,X) = \inf_{v\in U_2} \sup_{u \in U_1} H(t,x,p,X,u,v), \tag{6.29}$$

where

$$H(t,x,p,X,u,v) = L(t,x,u,v) + p^{\mathrm{T}}F(t,x,u,v) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(t,x,u,v)XD(t,x,u,v). \tag{6.30}$$

Equations (6.26) and (6.27) characterize the optimal control and stopper policies for a two-player stochastic time-varying differential game problem on a finite or infinite interval. For infinite horizon time-invariant differential games, with the controller constrained to a strictly progressively measurable strategy, the Isaacs minimax condition $V^-(t,x) = V^+(t,x) = V(x)$ holds, and hence, (6.26) and (6.27) reduce to (6.12)–(6.16).

Conditions (6.12)–(6.16) guarantee optimality with respect to the set of admissible stabilizing controllers $\mathcal{S}(x_0,\rho)$. However, it is important to note that an explicit characterization of the set $\mathcal{S}(x_0,\rho)$ is not required. In addition, the optimal stabilizing *feedback* control $u = \phi(x)$ and *feedback* stopper $v = \psi(x)$ laws are independent of the initial condition x_0. Finally, in order to ensure asymptotic stability in probability of the closed-loop system (6.1), Theorem 6.1 requires that $V(\cdot)$ satisfy (6.7), (6.8), and (6.11), which implies that $V(\cdot)$ is a Lyapunov function for the closed-loop system (6.1). However, for optimality $V(\cdot)$ need not satisfy (6.8) and (6.11). Specifically, if $V(\cdot)$ is a two-times continuously differentiable function such that (6.7) is satisfied and $\phi(\cdot) \in \mathcal{S}(x_0, \rho)$, then (6.12)–(6.16) imply (6.18) and (6.19).

The optimal feedback control and stopper policy $\phi(\cdot)$ and $\psi(\cdot)$ that guarantee global asymptotic stability in probability give $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot),\psi(\cdot)}\right) = 1$, and hence, $\mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot),\psi(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. Moreover, all the admissible control laws $u(\cdot)$ and stopper laws $v(\cdot)$ that guarantee global attraction in probability satisfy $\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}\right) = 1$ for all $x_0 \in \mathbb{R}^n$, and hence, $\rho = 0$ and $\mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \overset{\text{a.s.}}{=} 1$. In this case,

$$\begin{aligned} J\left(x_0, u(\cdot), v(\cdot)\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_0^\infty L(x(t), u(t), v(t)) \mathbb{1}_{\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}}(\omega) \mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_0^\infty L(x(t), u(t), v(t)) \mathrm{d}t\right] \end{aligned} \tag{6.31}$$

and

$$\begin{aligned} J\left(x_0, \phi(\cdot), \psi(\cdot)\right) &= \frac{1}{\mathbb{P}^{x_0}\left(\mathfrak{B}_{x_0}^{\phi(\cdot),\psi(\cdot)}\right)} \mathbb{E}^{x_0}\left[\int_0^\infty L(x(t), \phi(x(t)), \psi(x(t))) \mathbb{1}_{\mathfrak{B}_{x_0}^{\phi(\cdot),\psi(\cdot)}}(\omega) \mathrm{d}t\right] \\ &= \mathbb{E}^{x_0}\left[\int_0^\infty L(x(t), \phi(x(t)), \psi(x(t))) \mathrm{d}t\right]. \end{aligned} \tag{6.32}$$

Thus, in the remainder of the chapter, we omit the dependence on $\mathfrak{B}_{x_0}^{\phi(\cdot),\psi(\cdot)}$ and $\mathfrak{B}_{x_0}^{u(\cdot),v(\cdot)}$ in the cost functional and we write $\mathcal{S}(x_0)$ for $\mathcal{S}(x_0, \rho)$ for all the results concerning globally stabilizing controllers in probability.

Next, we specialize Theorem 6.1 to linear stochastic differential games and provide connections to the linear-quadratic stochastic differential game problem with multiplicative noise. For the following result let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_1}$, $C \in \mathbb{R}^{n\times m_2}$, $\sigma \in \mathbb{R}^d$, $R_1 \in \mathbb{P}^n$, $R_2 \in \mathbb{P}^{m_1}$, and $R_3 \in \mathbb{P}^{m_2}$ be given.

Corollary 6.1. Consider the linear-quadratic stochastic differential game problem with multiplicative noise given by

$$\mathrm{d}x(t) = \left[Ax(t) + Bu(t) + Cv(t)\right] \mathrm{d}t + x(t)\sigma^{\mathrm{T}} \mathrm{d}w(t), \qquad x(0) \overset{\text{a.s.}}{=} x_0, \qquad t \geq 0, \tag{6.33}$$

and with quadratic performance functional

$$J(x_0,u(\cdot),v(\cdot)) \triangleq \mathbb{E}^{x_0}\left[\int_0^\infty [x^{\mathrm{T}}(t)R_1x(t)+u^{\mathrm{T}}(t)R_2u(t)-v^{\mathrm{T}}(t)R_3v(t)]\mathrm{d}t\right], \tag{6.34}$$

where $u(\cdot)$ and $v(\cdot)$ are admissible and $u(\cdot)$ is constrained to a strictly progressively measurable strategy. Furthermore, assume that $BR_2^{-1}B^{\mathrm{T}} \geq CR_3^{-1}C^{\mathrm{T}}$ and there exists a positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$0=\left(A+\frac{1}{2}\|\sigma\|^2I_n\right)^{\mathrm{T}}P+P\left(A+\frac{1}{2}\|\sigma\|^2I_n\right)+R_1-PBR_2^{-1}B^{\mathrm{T}}P+PCR_3^{-1}C^{\mathrm{T}}P. \tag{6.35}$$

Then, with the feedback control law $u=\phi(x)\triangleq -R_2^{-1}B^{\mathrm{T}}Px$ and feedback stopper law $v=\psi(x)\triangleq R_3^{-1}C^{\mathrm{T}}Px$, the zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ to (6.33) is globally asymptotically stable in probability and

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot)))=x_0^{\mathrm{T}}Px_0, \qquad x_0\in\mathbb{R}^n. \tag{6.36}$$

Furthermore,

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot)))=\min_{u(\cdot)\in\mathcal{S}(x_0)}\max_{v\in\mathcal{U}_2}J(x_0,u(\cdot),v(\cdot)), \tag{6.37}$$

where $\mathcal{S}(x_0)$ is the set of regulation controllers for (6.33) and $x_0\in\mathbb{R}^n$.

Proof. The result is a direct consequence of Theorem 6.1 with $F(x,u,v)=Ax+Bu+Cv$, $D(x,u,v)=x\sigma^{\mathrm{T}}$, $L(x,u,v)=x^{\mathrm{T}}R_1x+u^{\mathrm{T}}R_2u-v^{\mathrm{T}}R_3v$, $V(x)=x^{\mathrm{T}}Px$, $\phi(x)=-R_2^{-1}B^{\mathrm{T}}Px$, $\psi(x)=R_3^{-1}C^{\mathrm{T}}Px$, $\mathcal{D}=\mathbb{R}^n$, $U_1=\mathbb{R}^{m_1}$, and $U_2=\mathbb{R}^{m_2}$. Specifically, first note that the controller and stopper policies are decoupled in the system dynamics and the payoff function, and hence, $u=\alpha(v)\triangleq f_u(x)$ and $v=\beta(u)\triangleq f_v(x)$. Thus, (6.15) and (6.16) are satisfied with $f_u(x)=\phi(x)$ and $f_v(x)=\psi(x)$. Now, conditions (6.7)–(6.10) are trivially satisfied.

Next, it follows from (6.35) that $H(x,\phi(x),\psi(x))=0$, and hence, $V'(x)F(x,\phi(x),\psi(x))+\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x,\phi(x),\psi(x))V''(x)D(x,\ \phi(x),\psi(x))<0$ for all $x\in\mathbb{R}^n$ and $x\neq 0$. Thus, $H(x,u,\psi(x))-H(x,\phi(x),\psi(x))=[u-\phi(x)]^{\mathrm{T}}R_2[u-\phi(x)]\geq 0$ and $H(x,\phi(x),v)-H(x,\phi(x),\psi(x))=-[v-\psi(x)]^{\mathrm{T}}R_3[v-\psi(x)]\leq 0$ so that all the conditions of Theorem 6.1 are satisfied. Finally, since $V(\cdot)$ is radially unbounded the zero solution $x(t)\overset{\text{a.s.}}{\equiv}0$ to (6.33), with $u(t)=$

$\phi(x(t)) = -R_2^{-1}B^{\mathrm{T}}Px(t)$ and $v(t) = \psi(x(t)) = R_3^{-1}C^{\mathrm{T}}Px(t)$, is globally asymptotically stable in probability. □

The optimal feedback control and stopper laws $\phi(x)$ and $\psi(x)$ in Corollary 6.1 are derived using the properties of $H(x,u,v)$ as defined in Theorem 6.1. Specifically, since $H(x,u,v) = x^{\mathrm{T}}R_1x + u^{\mathrm{T}}R_2u - v^{\mathrm{T}}R_3v + x^{\mathrm{T}}(A^{\mathrm{T}}P + PA)x + 2x^{\mathrm{T}}PBu + 2x^{\mathrm{T}}PCv + \|\sigma\|^2x^{\mathrm{T}}Px$ it follows that $\frac{\partial^2 H}{\partial u^2} = R_2 > 0$ and $\frac{\partial^2 H}{\partial v^2} = -R_3 < 0$. Now, $\frac{\partial H}{\partial u} = 2R_2u + 2B^{\mathrm{T}}Px = 0$ gives the unique global minimizer of $H(x,u,v)$, whereas $\frac{\partial H}{\partial v} = -2R_3v + 2C^{\mathrm{T}}Px = 0$ gives the unique global maximizer of $H(x,u,v)$. Hence, since $\phi(x)$ (resp., $\psi(x)$) minimizes (resp., maximizes) $H(x,u,v)$ it follows that $\phi(x)$ (resp., $\psi(x)$) satisfies $\frac{\partial H}{\partial u} = 0$ (resp., $\frac{\partial H}{\partial v} = 0$) or, equivalently, $\phi(x) = -R_2^{-1}B^{\mathrm{T}}Px$ (resp., $\psi(x) = -R_3^{-1}C^{\mathrm{T}}Px$).

Finally, we close this section by noting that the existence of a positive-definite solution P satisfying (6.35) can be guaranteed using $\mathcal{H}_\infty$ theory. Specifically, it follows from standard $\mathcal{H}_\infty$ theory [106] that (6.35) has a positive-definite solution if and only if the Hamiltonian matrix

$$\mathcal{H} = \begin{bmatrix} A + \frac{1}{2}\|\sigma\|^2 I_n & C^{\mathrm{T}}R_3^{-1}C - BR_2^{-1}B^{\mathrm{T}} \\ -R_1 & -\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} \end{bmatrix}$$

has no purely imaginary eigenvalues.

6.3. Inverse Optimal Stochastic Differential Games for Nonlinear Affine Systems

In this section, we specialize Theorem 6.1 to affine in the control and stopper differential game policies. Specifically, we construct nonlinear feedback controllers and stoppers using a stochastic differential game framework that minimizes and maximizes, respectively, a nonlinear-nonquadratic performance criterion. This is accomplished by choosing the controller and stopper policies such that the mapping of the infinitesimal generator of the Lyapunov function is negative along the closed-loop system trajectories while providing sufficient conditions for the existence of asymptotically stabilizing (in probability) solutions to the stochastic Hamilton-Jacobi-Isaacs equation. Thus, these results provide a family of globally stabilizing controller and stopper policies parameterized by the cost functional that is optimized.

The controller and stopper policies obtained in this section are predicated on an *inverse optimal stochastic differential game problem*. The related inverse optimal control problem is discussed in Chapter 3. In particular, to avoid the complexity in solving the stochastic steady-state Hamilton-Jacobi-Isaacs equation we do not attempt to optimize a *given* cost functional, but rather, we parameterize a family of stochastically stabilizing controllers that optimize some *derived* cost functional that provides flexibility in specifying the control and stopper policies. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function for the closed-loop system, and the stabilizing feedback control law as well as the stopper law, wherein the coupling is introduced via the stochastic Hamilton-Jacobi-Isaacs equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally stabilizing in probability controllers that can meet closed-loop system response constraints.

Consider the nonlinear affine in the control and stopper two-player stochastic differential game problem given by

$$\mathrm{d}x(t) = [f(x(t)) + G_1(x(t))u(t) + G_2(x(t))v(t)]\,\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.38}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(0) = 0$, $G_1 : \mathbb{R}^n \to \mathbb{R}^{n\times m_1}$, $G_2 : \mathbb{R}^n \to \mathbb{R}^{n\times m_2}$, $D : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ satisfies $D(0) = 0$, $\mathcal{D} = \mathbb{R}^n$, $U_1 = \mathbb{R}^{m_1}$, and $U_2 = \mathbb{R}^{m_2}$. Furthermore, we consider performance integrands $L(x, u, v)$ of the form

$$L(x, u, v) = L_1(x) + L_{2u}(x)u + u^{\mathrm{T}}R_2(x)u - L_{2v}(x)v - v^{\mathrm{T}}R_3(x)v, \tag{6.39}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_{2u} : \mathbb{R}^n \to \mathbb{R}^{1\times m_1}$, $L_{2v} : \mathbb{R}^n \to \mathbb{R}^{1\times m_2}$, $R_2 : \mathbb{R}^n \to \mathbb{P}^{m_1}$, and $R_3 : \mathbb{R}^n \to$

$\mathbb{P}^{m_2}$ so that (6.3) becomes

$$J(x_0, u(\cdot), v(\cdot)) = \mathbb{E}^{x_0}\Bigg[\int_0^\infty [L_1(x(t)) + L_{2u}(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x(t))u(t) \\ - L_{2v}(x(t))v(t) - v^{\mathrm{T}}(t)R_3(x(t))v(t)]\mathrm{d}t\Bigg]. \tag{6.40}$$

Theorem 6.2. Consider the nonlinear two-player stochastic differential game problem (6.38) with performance functional (6.40) where $u(\cdot)$ and $v(\cdot)$ are admissible and $u(\cdot)$ is constrained to a progressively measurable strategy. Assume that there exist a two-times continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ and functions $L_{2u} : \mathbb{R}^n \to \mathbb{R}^{1\times m_1}$ and $L_{2v} : \mathbb{R}^n \to \mathbb{R}^{1\times m_2}$ such that

$$V(0) = 0, \tag{6.41}$$

$$L_{2u}(0) = 0, \tag{6.42}$$

$$L_{2v}(0) = 0, \tag{6.43}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n, \qquad x \neq 0, \tag{6.44}$$

$$V'(x)\Bigg[f(x) - \frac{1}{2}G_1(x)R_2^{-1}(x)L_{2u}^{\mathrm{T}}(x) - \frac{1}{2}G_1(x)R_2^{-1}(x)G_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) - \frac{1}{2}G_2(x)R_3^{-1}(x)L_{2v}^{\mathrm{T}}(x) \\ +\frac{1}{2}G_2(x)R_3^{-1}(x)G_2^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\Bigg] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0, \quad x \in \mathbb{R}^n, \quad x \neq 0, \tag{6.45}$$

and

$$V(x) \to \infty \text{ as } \|x\| \to \infty. \tag{6.46}$$

Then, the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = [f(x(t))+G_1(x(t))\phi(x(t))+G_2(x(t))\psi(x(t))]\mathrm{d}t+D(x(t))\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.47}$$

is globally asymptotically stable in probability with the feedback control and stopper laws

$$\phi(x) = -\tfrac{1}{2}R_2^{-1}(x)[V'(x)G_1(x) + L_{2u}(x)]^{\mathrm{T}}, \tag{6.48}$$

$$\psi(x) = \tfrac{1}{2}R_3^{-1}(x)[V'(x)G_2(x) - L_{2v}(x)]^{\mathrm{T}}, \tag{6.49}$$

and the performance functional (6.40), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - \psi^{\mathrm{T}}(x)R_3(x)\psi(x) - V'(x)f(x) - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x), \quad (6.50)$$

is optimized in the sense that

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} \max_{v(\cdot)\in\mathcal{U}_2} J(x_0, u(\cdot), v(\cdot)), \qquad x_0 \in \mathbb{R}^n. \quad (6.51)$$

Finally,

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \quad (6.52)$$

Proof. The result is a direct consequence of Theorem 6.1 with $\mathcal{D} = \mathbb{R}^n$, $U_1 = \mathbb{R}^{m_1}$, $U_2 = \mathbb{R}^{m_2}$, $F(x,u,v) = f(x) + G_1(x)u + G_2(x)v$, $D(x,u,v) = D(x)$, and $L(x,u,v) = L_1(x) + L_{2u}(x)u + u^{\mathrm{T}}R_2(x)u - L_{2v}(x)v - v^{\mathrm{T}}R_3(x)v$. Specifically, with (6.39) the Hamiltonian has the form

$$\begin{aligned} H(x,u,v) = \ & L_1(x) + L_{2u}(x)u + u^{\mathrm{T}}R_2(x)u - L_{2v}(x)v - v^{\mathrm{T}}R_3(x)v \\ & + V'(x)(f(x) + G_1(x)u + G_2(x)v) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x). \end{aligned}$$

Now, the feedback control and stopper laws (6.48) and (6.49) are obtained by setting $\frac{\partial H}{\partial u} = 0$ and $\frac{\partial H}{\partial v} = 0$. With (6.48) and (6.49), it follows that (6.41), (6.44), (6.45), and (6.46) imply (6.7), (6.8), (6.11), and (6.20), respectively. Furthermore, since the controller and stopper are decoupled in the cost and system dynamics, it follows from the information neutrality that the optimal strategy for the controller is given by $\alpha(v) = \phi(x)$, $v \in U_2$, and the optimal strategy for the stopper is given by $\beta(u) = \psi(x)$, $u \in U_1$. Thus, conditions (6.15) and (6.16) are trivially satisfied.

Next, since $V(\cdot)$ is two-times continuously differentiable and $x = 0$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$, and hence, since by assumption $L_{2u}(0) = 0$ and $L_{2v}(0) = 0$, it follows that $\phi(0) = 0$ and $\psi(0) = 0$, which implies (6.9) and (6.10). Next, with $L_1(x)$ given by (6.50) and $\phi(x)$ and $\psi(x)$ given by (6.48) and (6.49), respectively, (6.12) holds. Finally, since $H(x,u,\psi(x)) - H(x,\phi(x),\psi(x)) = [u - \phi(x)]^{\mathrm{T}}R_2[u - \phi(x)]$ and $H(x,\phi(x),v) -$

$H(x,\phi(x),\psi(x)) = -[v-\psi(x)]^{\mathrm{T}}R_3[v-\psi(x)]$, and $R_2(x)$ and $R_3(x)$ are positive definite for all $x \in \mathbb{R}^n$, conditions (6.13) and (6.14) hold. The result now follows as a direct consequence of Theorem 6.1. □

Note that (6.45) is equivalent to

$$\mathcal{L}V(x) \triangleq V'(x)[f(x)+G_1(x)\phi(x)+G_2(x)\psi(x)]+\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) < 0,\ \ x \in \mathbb{R}^n,\ \ x \neq 0, \tag{6.53}$$

with $\phi(x)$ and $\psi(x)$ given by (6.48) and (6.49), respectively. Furthermore, conditions (6.41), (6.44), and (6.53) ensure that $V(\cdot)$ is a Lyapunov function for the closed-loop system (6.47). As discussed in [45], it is important to recognize that the functions $L_{2u}(x)$ and $L_{2v}(x)$, which appear in the integrand of the performance functional (6.39), are an arbitrary function of $x \in \mathbb{R}^n$ subject to conditions (6.42), (6.43) and (6.45). Thus, $L_{2u}(x)$ and $L_{2v}(x)$ provide flexibility in choosing the control and stopper policies.

With $L_1(x)$ given by (6.50) and $\phi(x)$ and $\psi(x)$ given by (6.48) and (6.49), repectively, $L(x,u,v)$ can be expressed as

$$\begin{aligned}
L(x,u,v) \ =\ & u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - v^{\mathrm{T}}R_3(x)v + \psi^{\mathrm{T}}(x)R_3(x)\psi(x) \\
& + L_{2u}(x)(u-\phi(x)) - L_{2v}(x)(v-\psi(x)) - V'(x)[f(x)+G_1(x)\phi(x) \\
& + G_2(x)\psi(x)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\
=\ & \left[u+\tfrac{1}{2}R_2^{-1}(x)L_{2u}^{\mathrm{T}}(x)\right]^{\mathrm{T}}R_2(x)\left[u+\tfrac{1}{2}R_2^{-1}(x)L_{2u}^{\mathrm{T}}(x)\right] \\
& - \left[v+\tfrac{1}{2}R_3^{-1}(x)L_{2v}^{\mathrm{T}}(x)\right]^{\mathrm{T}}R_3(x)\left[v+\tfrac{1}{2}R_3^{-1}(x)L_{2v}^{\mathrm{T}}(x)\right] \\
& - V'(x)[f(x)+G_1(x)\phi(x)+G_2(x)\psi(x)] - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\
& - \tfrac{1}{4}V'(x)G_1(x)R_2^{-1}(x)G_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) + \tfrac{1}{4}V'(x)G_2(x)R_3^{-1}(x)G_2^{\mathrm{T}}(x)V'^{\mathrm{T}}(x).
\end{aligned} \tag{6.54}$$

Since $R_2(x) > 0$ and $R_3(x) > 0$, $x \in \mathbb{R}^n$, the first and last term on the right-hand side of (6.54) is nonnegative, while (6.53) implies that the third, fourth, fifth, and sixth terms collectively are nonnegative. Thus, it follows that

$$L(x,u,v) \geq -\left[v+\tfrac{1}{2}R_3^{-1}(x)L_{2v}^{\mathrm{T}}(x)\right]^{\mathrm{T}}R_3(x)\left[v+\tfrac{1}{2}R_3^{-1}(x)L_{2v}^{\mathrm{T}}(x)\right]$$

$$-\tfrac{1}{4}V'(x)G_1(x)R_2^{-1}(x)G_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \tag{6.55}$$

which shows that $L(x,u,v)$ may be negative. As a result, there may exist control and stopper policies u and v for which the performance functional $J(x_0,u,v)$ is negative. However, if the control u is a regulation controller, that is, $u \in \mathcal{S}(x_0)$, then it follows from (6.51) and (6.52) that

$$J(x_0,u(\cdot),v(\cdot)) \geq V(x_0) \geq 0, \qquad x_0 \in \mathbb{R}^n, \qquad u(\cdot) \in \mathcal{S}(x_0). \tag{6.56}$$

Next, we specialize Theorem 6.2 to linear stochastic differential games characterized by nonlinear controllers and stoppers that, respectively, minimize and maximize a polynomial cost functional. For the following result let $\sigma \in \mathbb{R}^d$, $R_1 \in \mathbb{P}^n$, $R_2 \in \mathbb{P}^{m_1}$, $R_3 \in \mathbb{P}^{m_2}$, and $\hat{R}_q \in \mathbb{N}^n$, $q = 2,\ldots,r$, be given, where r is a positive integer, and define $S_1 \triangleq BR_2^{-1}B^{\mathrm{T}}$, $S_2 \triangleq CR_3^{-1}C^{\mathrm{T}}$, and $S \triangleq S_1 - S_2$.

Corollary 6.2. Consider the two-player stochastic differential game problem with multiplicative noise given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t) + Cv(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\mathrm{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.57}$$

where $u(\cdot)$ and $v(\cdot)$ are admissible and $u(\cdot)$ is constrained to a strictly progressively measurable strategy. Assume that S is nonnegative definite and there exist $P \in \mathbb{P}^n$ and $M_q \in \mathbb{N}^n$, $q = 2,\ldots,r$, such that

$$0 = \left(A + \frac{1}{2}\|\sigma\|^2 I_n\right)^{\mathrm{T}} P + P\left(A + \frac{1}{2}\|\sigma\|^2 I_n\right) + R_1 - PSP, \tag{6.58}$$

$$0 = \left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right)^{\mathrm{T}} M_q + M_q\left(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\right) + \hat{R}_q,$$
$$q = 2,\ldots,r. \tag{6.59}$$

Then, the zero solution $x(t) \overset{\mathrm{a.s.}}{\equiv} 0$ of the closed-loop system

$$\mathrm{d}x(t) = (Ax(t) + B\phi(x(t)) + C\psi(x(t)))\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\mathrm{a.s.}}{=} x_0, \quad t \geq 0, \tag{6.60}$$

is globally asymptotically stable in probability with the feedback control and stopper policies

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}\left(P+\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right)x, \tag{6.61}$$

$$\psi(x) = R_3^{-1}C^{\mathrm{T}}\left(P+\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right)x, \tag{6.62}$$

and the performance functional (6.40), with $R_2(x) = R_2$, $R_3(x) = R_3$, $L_{2u}(x) = 0$, $L_{2v}(x) = 0$, and

$$\begin{aligned} L_1(x) = x^{\mathrm{T}}\Bigg(&R_1+\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}\hat{R}_q+\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]^{\mathrm{T}}S \\ &\cdot\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]\Bigg)x, \end{aligned} \tag{6.63}$$

is optimized in the sense that

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)}\max_{v(\cdot)\in\mathcal{U}_2} J(x_0,u(\cdot),v(\cdot)), \qquad x_0\in\mathbb{R}^n. \tag{6.64}$$

Finally,

$$J(x_0,\phi(x(\cdot)),\psi(x(\cdot))) = x_0^{\mathrm{T}}Px_0+\sum_{q=2}^{r}\tfrac{1}{q}(x_0^{\mathrm{T}}M_qx_0)^q, \qquad x_0\in\mathbb{R}^n. \tag{6.65}$$

Proof. The result is a direct consequence of Theorem 6.2 with $f(x) = Ax$, $G_1(x) = B$, $G_2(x) = C$, $D(x) = x\sigma^{\mathrm{T}}$, $L_{2u}(x) = 0$, $L_{2v}(x) = 0$, $R_2(x) = R_2$, $R_3(x) = R_3$, and

$$V(x) = x^{\mathrm{T}}Px+\sum_{q=2}^{r}\frac{1}{q}(x^{\mathrm{T}}M_qx)^q.$$

Specifically, (6.41)–(6.44) and (6.46) are trivially satisfied. Next, it follows from (6.58), (6.59), and (6.61) that

$$\begin{aligned} &V'(x)[f(x)-\tfrac{1}{2}G_1(x)R_2^{-1}(x)G_1^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)+\tfrac{1}{2}G_2(x)R_3^{-1}(x)G_2^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)] \\ &\quad+\frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ &\qquad\qquad = -x^{\mathrm{T}}R_1x-\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}x^{\mathrm{T}}\hat{R}_qx-\phi^{\mathrm{T}}(x)R_2\phi(x) \end{aligned}$$

$$
+\psi^{\mathrm{T}}(x)R_3\psi(x) - x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]^{\mathrm{T}} S\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]x,
$$

$$
\begin{aligned}
&= -x^{\mathrm{T}}R_1x - \sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}x^{\mathrm{T}}\hat{R}_qx \\
&\quad -x^{\mathrm{T}}\left(P+\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right)S\left(P+\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right)x \\
&\quad -x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]^{\mathrm{T}} S\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_qx)^{q-1}M_q\right]x,
\end{aligned}
$$

which implies (6.45), so that all the conditions of Theorem 6.2 are satisfied. □

Corollary 6.2 requires the solutions of $r-1$ modified Riccati equations in (6.59) to obtain the optimal controller and stopper policies (6.61) and (6.62), respectively. It is important to note that the derived performance functional weighs the state variables by arbitrary even powers.

Next, we specialize Theorem 6.2 to linear stochastic differential games characterized by nonlinear controller and stopper policies that, respectively, minimize and maximize a multilinear cost functional. For the following result recall the definitions of $x^{[q]} \triangleq x\otimes x\otimes\cdots\otimes x$ and $\overset{q}{\oplus}\, A \triangleq A\oplus A\oplus\cdots\oplus A$, where x and A appear q times and q is positive integer. Furthermore, recall the definition of S and let $R_1\in\mathbb{P}^n$, $R_2\in\mathbb{P}^{m_1}$, $R_3\in\mathbb{P}^{m_2}$, and $\hat{R}_{2q}\in\mathcal{N}^{(2q,n)}$, $q=2,\ldots,r$, be given, where r is a given integer and $\mathcal{N}^{(k,n)}\triangleq\{\Psi\in\mathbb{R}^{1\times n^k}:\Psi x^{[k]}\geq 0, x\in\mathbb{R}^n\}$.

Corollary 6.3. Consider the two-player stochastic differential game problem (6.57) where $u(\cdot)$ and $v(\cdot)$ are admissible and $u(\cdot)$ is constrained to a strictly progressively measurable strategy. Assume that S is nonnegative definite and there exist $P\in\mathbb{P}^n$ and $\hat{P}_q\in\mathcal{N}^{(2q,n)}$, $q=2,\ldots,r$, such that

$$
0 = \left(A+\frac{1}{2}\|\sigma\|^2I_n\right)^{\mathrm{T}}P + P\left(A+\frac{1}{2}\|\sigma\|^2I_n\right) + R_1 - PSP, \tag{6.66}
$$

$$
0 = \hat{P}_q\left[\overset{2q}{\oplus}\left(A+\frac{1}{2}(2q-1)\|\sigma\|^2I_n - SP\right)\right] + \hat{R}_{2q}, \qquad q=2,\ldots,r. \tag{6.67}
$$

Then the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop system (6.60) is globally asymptotically stable in probability with the feedback control and stopper policies

$$\phi(x) = -R_2^{-1}B^{\mathrm{T}}(Px + \tfrac{1}{2}g'^{\mathrm{T}}(x)), \tag{6.68}$$

$$\psi(x) = R_3^{-1}C^{\mathrm{T}}(Px + \tfrac{1}{2}g'^{\mathrm{T}}(x)), \tag{6.69}$$

where $g(x) \triangleq \sum_{q=2}^{r} \hat{P}_q x^{[2q]}$, and the performance functional (6.40) with $R_2(x) = R_2$, $R_3(x) = R_3$, $L_{2u}(x) = 0$, $L_{2v}(x) = 0$, and

$$L_1(x) = x^{\mathrm{T}}R_1x + \sum_{q=2}^{r} \hat{R}_{2q}x^{[2q]} + \tfrac{1}{4}g'(x)Sg'^{\mathrm{T}}(x), \tag{6.70}$$

is minimized in the sense of (6.51). Finally,

$$J(x_0, \phi(x(\cdot))) = x_0^{\mathrm{T}}Px_0 + \sum_{q=2}^{r} \hat{P}_q x_0^{[2q]}, \qquad x_0 \in \mathbb{R}^n. \tag{6.71}$$

Proof. The result is a direct consequence of Theorem 6.2 with $f(x) = Ax$, $G_1(x) = B$, $G_2(x) = C$, $D(x) = x\sigma^{\mathrm{T}}$, $L_{2u}(x) = 0$, $L_{2v}(x) = 0$, $R_2(x) = R_2$, $R_3(x) = R_3$, and $V(x) = x^{\mathrm{T}}Px + \sum_{q=2}^{r} \hat{P}_q x^{[2q]}$. Specifically, (6.41)–(6.44) are trivially satisfied. Next, it follows from (6.66)–(6.69) that

$$\begin{aligned}
&V'(x)[f(x) - \tfrac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) \\
&\quad + \tfrac{1}{2}G_2(x)R_3^{-1}(x)G_2^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\
&\qquad = -x^{\mathrm{T}}R_1x - \sum_{q=2}^{r} \hat{R}_{2q}x^{[2q]} - \phi^{\mathrm{T}}(x)R_2\phi(x) + \psi^{\mathrm{T}}(x)R_3\psi(x) \\
&\qquad\quad -\tfrac{1}{4}g'(x)S_1g'^{\mathrm{T}}(x) + \tfrac{1}{4}g'(x)S_2g'^{\mathrm{T}}(x) \\
&\qquad = -x^{\mathrm{T}}R_1x - \sum_{q=2}^{r} \hat{R}_{2q}x^{[2q]} - (Px + \tfrac{1}{2}g'^{\mathrm{T}}(x))^{\mathrm{T}}S(Px + \tfrac{1}{2}g'^{\mathrm{T}}(x)) - \tfrac{1}{4}g'(x)Sg'^{\mathrm{T}}(x)
\end{aligned}$$

which implies (6.45) so that all the conditions of Theorem 6.2 are satisfied. □

Note that since

$$g'(x)(A - SP)x + \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}}g''(x)(x\sigma^{\mathrm{T}}) = \sum_{q=2}^{r} \hat{P}_q \Big[\overset{2q}{\oplus} \big(A + \frac{1}{2}(2q-1)\|\sigma\|^2 I_n - SP\big)\Big] x^{[2q]},$$

it follows that (6.67) can be equivalently written as

$$0 = \frac{1}{2}\mathrm{tr}(x\sigma^{\mathrm{T}})^{\mathrm{T}} g''(x)(x\sigma^{\mathrm{T}}) + g'(x)(A - SP)x + \sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]}, \quad x \in \mathbb{R}^n,$$

and hence, it follows from Lemma 3.1, with A and $h(x)$ replaced by $(A - SP)$ and $\sum_{q=2}^{r} \hat{R}_{2q} x^{[2q]}$, respectively, that there exists a unique $\hat{P}_q \in \mathcal{N}^{(2q,n)}$ such that (6.67) is satisfied.

6.4. Illustrative Numerical Examples

In this section, we present two numerical examples to demonstrate the efficacy of the proposed differential game framework.

Example 6.1. In this example, we seek a stabilizing control policy of a quadrotor helicopter (i.e., a quadcopter) subject to an adversary stopper policy and a stochastic state disturbance. The coordinate systems and free body diagram for the quadcopter are shown in Figure 6.1. Specifically, the inertial frame is defined by the axes x_{I}, y_{I}, and z_{I}, and the body frame B is attached to the quadcopter with the x_{B} axis denoting the forward flight direction and the z_{B} axis denoting the perpendicular direction to the plane of the rotors with an upward orientation corresponding to perfect hover.

The linearized quadcopter dynamics about a perfect hover equilibrium point [104] are given by

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{r}_x(t) \\ \ddot{r}_y(t) \\ \ddot{r}_z(t) \end{bmatrix} = \begin{bmatrix} 0 & -mg & 0 \\ mg & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi(t) \\ \theta(t) \\ \sum_{i=1}^{4} T_i(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}, \quad t \geq 0, \tag{6.72}$$

$$\begin{bmatrix} I_{x_B} & 0 & 0 \\ 0 & I_{y_B} & 0 \\ 0 & 0 & I_{z_B} \end{bmatrix} \begin{bmatrix} \ddot{\phi}(t) \\ \ddot{\theta}(t) \\ \ddot{\psi}(t) \end{bmatrix} = \begin{bmatrix} 0 & \ell & 0 & -\ell \\ \ell & 0 & -\ell & 0 \\ k_r & -k_r & k_r & -k_r \end{bmatrix} \begin{bmatrix} T_1(t) \\ T_2(t) \\ T_3(t) \\ T_4(t) \end{bmatrix}, \tag{6.73}$$

where, for every $t \geq 0$, $r_x(t)$, $r_y(t)$, and $r_z(t)$ denote the position of the quadcopter, $\phi(t)$, $\theta(t)$, and $\psi(t)$ denote the angular position of the quadcopter, $m = 0.5\,\mathrm{kg}$ is the mass of the quadcopter, $g = 9.81\,\mathrm{m/sec}^2$ is the gravitational constant, $\ell = 0.17\,\mathrm{m}$ is the moment arm from

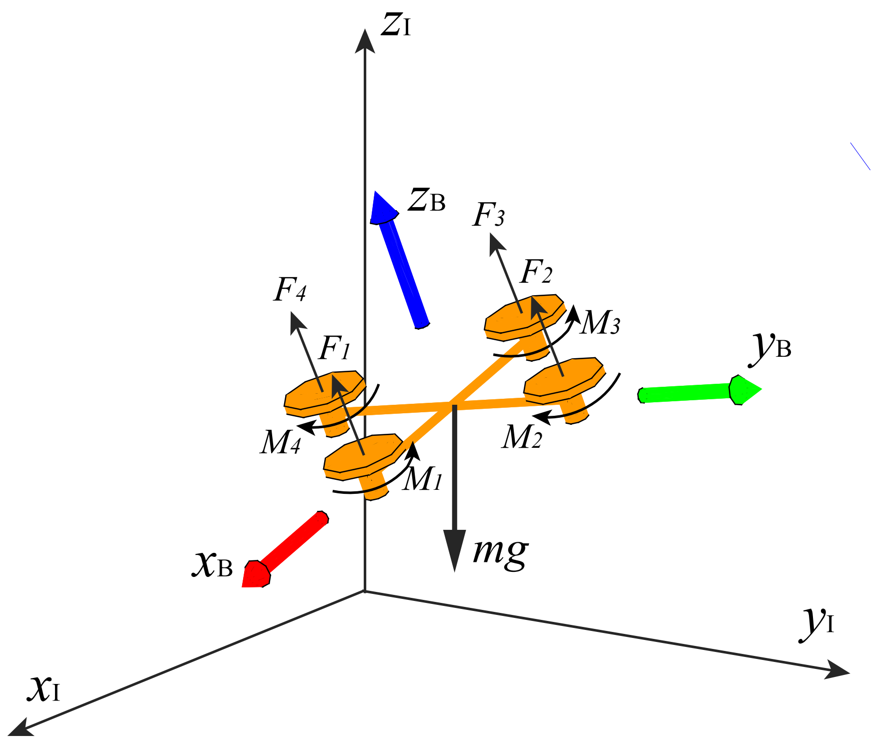

Figure 6.1: Coordinate systems and forces/moments acting on a quadcopter.

the body of the quadcopter to the motor, $I = \mathrm{diag}[I_{x_B}, I_{y_B}, I_{z_B}]$ is the mass moment of inertia matrix for the quadcopter in the body frame, with $I_{x_B} = 0.0036\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_{y_B} = 0.0036\,\mathrm{kg}\cdot\mathrm{m}^2$, and $I_{z_B} = 0.0070\,\mathrm{kg}\cdot\mathrm{m}^2$, and $T_i(t)$, $t \geq 0$, $i = 1, \ldots, 4$, denotes the thrusts generating the torques $M_i(t)$, $t \geq 0$, $i = 1, \ldots, 4$, where $M_i(t) = k_r T_i(t)$ and $k_r = 0.0245\,\mathrm{m}$.

Formulating (6.72) and (6.73) as a differential game problem gives (6.33) where the state vector is given by $x(t) = [r^{\mathrm{T}}(t), \Phi^{\mathrm{T}}(t), \dot{r}^{\mathrm{T}}(t), \dot{\Phi}^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{12}$, where $r(t) = [r_x(t), r_y(t), r_z(t)]^{\mathrm{T}}$ and $\Phi(t) = [\phi(t), \theta(t), \psi(t)]^{\mathrm{T}}$, the control is given by $u(t) = [T_1(t) - g/4, T_2(t) - g/4, T_3(t) - g/4, T_4(t) - g/4]^{\mathrm{T}}$, and the stopper is given by $v(t) = [v_1(t), v_2(t), v_3(t), v_4(t)]^{\mathrm{T}}$. Note that the control $u(t)$, $t \geq 0$, compensates for the constant offset term appearing in the dynamics of $\ddot{r}_z(t)$, $t \geq 0$, in (6.73). Here, $w(\cdot)$ is a one-dimensional standard Wiener process with variance $\sigma = 0.5$. Furthermore, $A \in \mathbb{R}^{16\times 16}$ is such that $A_{(1,7)} = A_{(2,8)} = A_{(3,9)} = A_{(4,10)} = A_{(5,11)} = A_{(6,12)} = 1$, $A_{(7,5)} = -g$, and $A_{(8,4)} = g$, and with all the other entries in A being zero. Finally, $B \in \mathbb{R}^{12\times 4}$ and $C \in \mathbb{R}^{12\times 4}$ are given by

$$B = C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & \frac{1}{m} & \frac{1}{m} & \frac{1}{m} \\ 0 & \frac{\ell}{I_{x_B}} & 0 & -\frac{\ell}{I_{x_B}} \\ \frac{\ell}{I_{y_B}} & 0 & -\frac{\ell}{I_{y_B}} & 0 \\ \frac{k_r}{I_{z_B}} & -\frac{k_r}{I_{z_B}} & \frac{k_r}{I_{z_B}} & -\frac{k_r}{I_{z_B}} \end{bmatrix}. \tag{6.74}$$

Here we design an optimal minimax solution using Corollary 6.1 with $x_0 = [0.4,\ -0.2,\ 0.1,\ 0.1,\ 0.1,\ -0.1,\ 0.05,\ -0.05,\ 0.05,\ -0.05,\ 0.05,\ -0.05]^{\mathrm{T}}$, $R_2 = 0.1I_{12}$, and $R_3 = I_{12}$. Figure 6.2 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figures 6.3 and 6.4 show the sample average along with the standard deviation of the corresponding control and stopper signals versus time for 20 sample paths, respectively. △

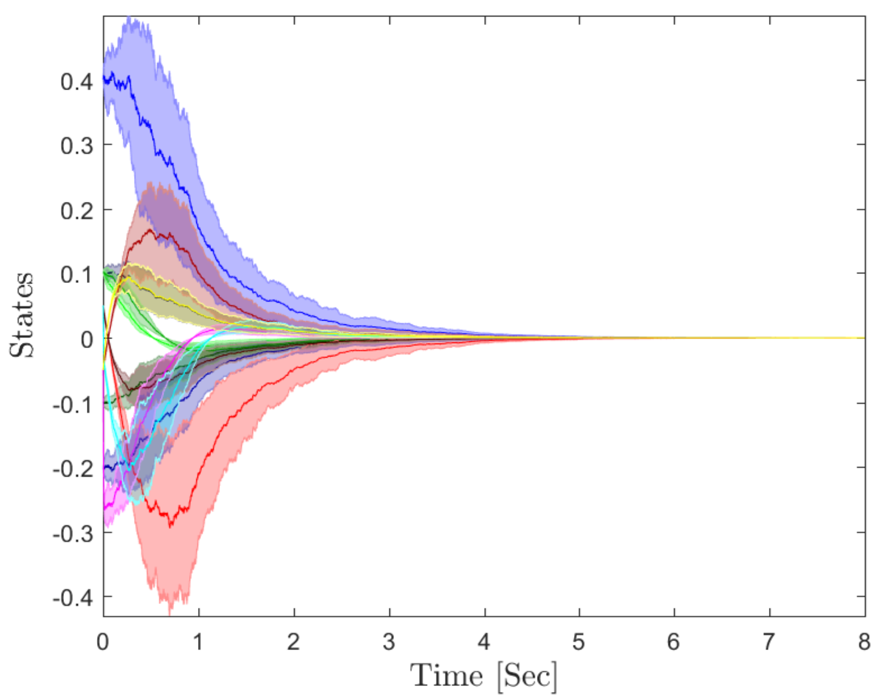

Figure 6.2: Sample average along with the sample standard deviation of the closed-loop system trajectory versus time.

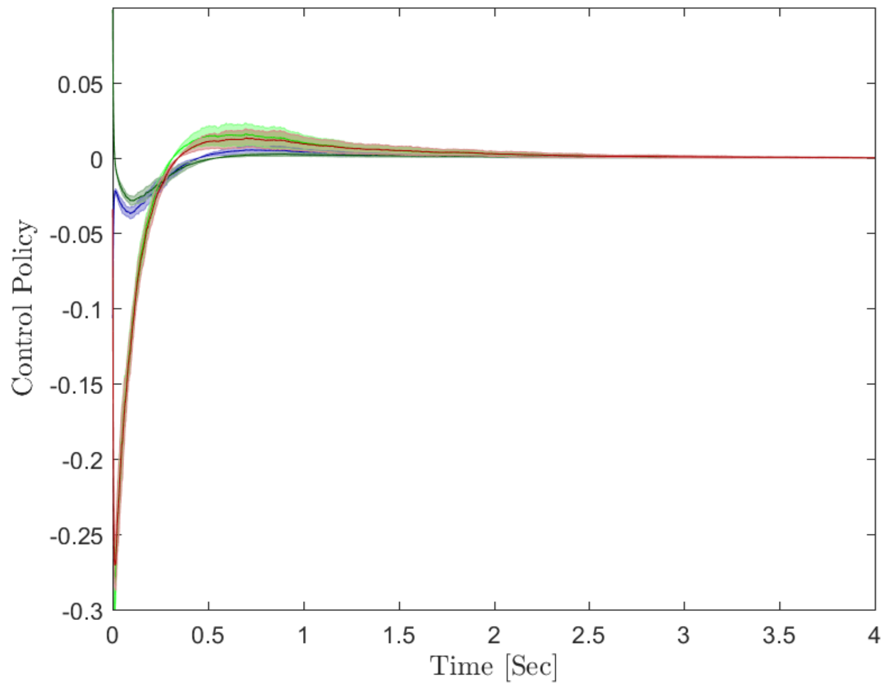

Figure 6.3: Sample average along with the sample standard deviation of the control signal versus time.

Example 6.2. Consider the nonlinear affine in the control and stopper stochastic differential game problem given by (6.38) capturing the inverted pendulum dynamics shown in Figure 6.5 with states $x(t) = [\theta(t), \dot{\theta}(t)]^{\mathrm{T}} \in \mathbb{R}^2$ and control and stopper policies $u(t) \in \mathbb{R}$, $t \geq 0$, and $v(t) \in \mathbb{R}$, $t \geq 0$. Here, $w(\cdot)$ denotes a one-dimensional standard Wiener process with variance $\sigma = 0.3$. In this case,

$$f(x) = \begin{bmatrix} \dot{\theta} \\ mgL\sin\theta/I \end{bmatrix}, \quad G_1(x) = G_2(x) = G = \begin{bmatrix} 0 \\ 1/I \end{bmatrix}, \quad D(x) = \sigma \begin{bmatrix} 0 \\ \dot{\theta} \end{bmatrix}, \tag{6.75}$$

where m, g, L, and I are constant with $mgL = 1\,\mathrm{N}\cdot\mathrm{m}$ and $I = 1\,\mathrm{kg}\cdot\mathrm{m}^2$. For this problem we consider the performance measure

$$J(x_0, u(\cdot), v(\cdot)) = \mathbb{E}^{x_0}\left[\int_0^\infty [L_1(x(t)) + u^{\mathrm{T}}(t)R_2u(t) - v^{\mathrm{T}}(t)R_3v(t)]\mathrm{d}t\right], \tag{6.76}$$

with $R_2 = 0.25$ and $R_3 = 1$.

Here we use Theorem 6.2 to construct an inverse optimal globally stabilizing controller and stopper policies for the inverted pendulum problem. Consider the Lyapunov function

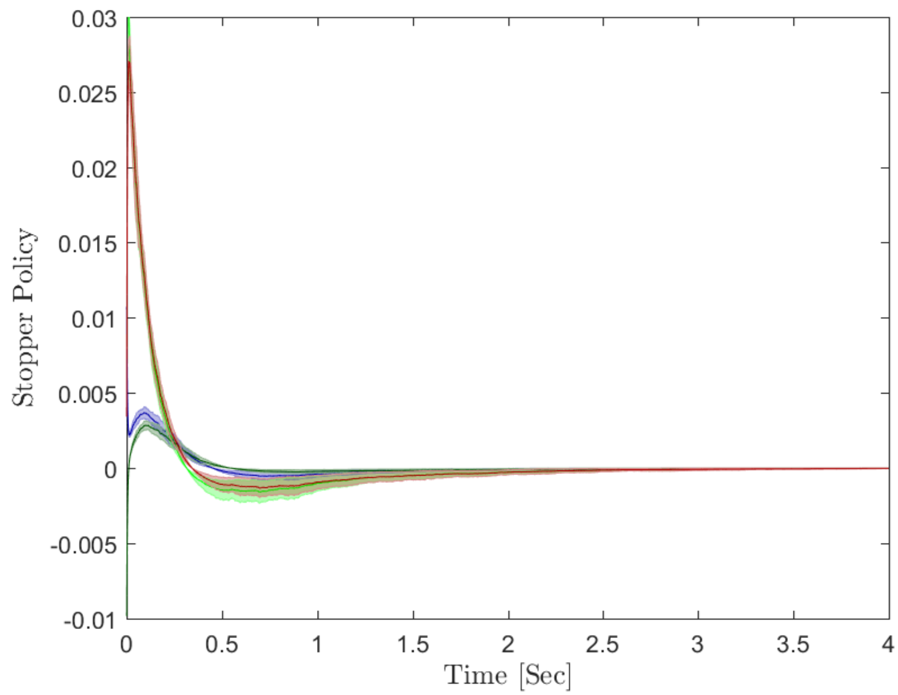

Figure 6.4: Sample average along with the sample standard deviation of the stopper signal versus time.

candidate given by

$$V(x) = \frac{1}{2}\dot{\theta}^2 + \alpha\left(\theta\dot{\theta} + \frac{1}{2}\theta^2\right) + \frac{1}{2}\theta^2 - (1 - \cos\theta), \tag{6.77}$$

where $\alpha \in (0, 1]$, and note that $\frac{1}{2}\theta^2 - (1 - \cos\theta) \geq 0$, $\theta \in \mathbb{R}$, with equality holding only if $\theta = 0$. Moreover, note that

$$\frac{1}{2}\dot{\theta}^2 + \alpha\left(\theta\dot{\theta} + \frac{1}{2}\theta^2\right) = \frac{1-\alpha}{2}\dot{\theta}^2 + \frac{\alpha}{2}(\theta + \dot{\theta})^2 \geq 0, \quad (\theta, \dot{\theta}) \in \mathbb{R} \times \mathbb{R}, \quad \alpha \in (0, 1],$$

and hence, $V(x) > 0$, $x \in \mathbb{R}^2$, $x \neq 0$, and $V(x) \to \infty$ as $\|x\| \to \infty$. Finally, using $\theta \sin\theta \leq \theta^2$ note that

$$\begin{aligned} V'(x)\left[f(x) - \frac{1}{2}G_1 R_2^{-1} G_1^{\mathrm{T}} V'^{\mathrm{T}}(x) + \frac{1}{2}G_2 R_3^{-1} G_2^{\mathrm{T}} V'^{\mathrm{T}}(x)\right] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x) V''(x) D(x) \\ = (1+\alpha)\theta\dot{\theta} + \alpha\dot{\theta}^2 + \alpha\theta\sin\theta - \frac{R_2^{-1}}{2}(\dot{\theta} + \alpha\theta)^2 + \frac{R_3^{-1}}{2}(\dot{\theta} + \alpha\theta)^2 + \frac{\sigma^2}{2}\dot{\theta}^2 \\ \leq \left(\alpha - \frac{r\alpha^2}{2}\right)\theta^2 + (1 + \alpha - r\alpha)\theta\dot{\theta} + \left(\alpha - \frac{r}{2} + \frac{\sigma^2}{2}\right)\dot{\theta}^2, \quad (\theta, \dot{\theta}) \in \mathbb{R} \times \mathbb{R}, \end{aligned} \tag{6.78}$$

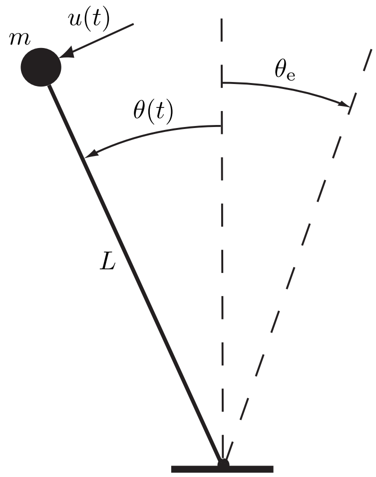

Figure 6.5: Inverted Pendulum.

where $r \triangleq R_2^{-1} - R_3^{-1}$.

With $\alpha = 0.9$, (6.78) yields

$$\begin{aligned} V'(x)\left[f(x) - \frac{1}{2}G_1 R_2^{-1} G_1^{\mathrm{T}} V'^{\mathrm{T}}(x) + \frac{1}{2}G_2 R_3^{-1} G_2^{\mathrm{T}} V'^{\mathrm{T}}(x)\right] + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x) \\ \leq -0.315\theta^2 - 0.8\theta\dot{\theta} - 0.555\dot{\theta}^2 \\ = -0.015\theta^2 - 0.3\left(\theta + \frac{4}{3}\dot{\theta}\right)^2 - 0.0216\dot{\theta}^2 \\ < 0, \quad (\theta, \dot{\theta}) \neq (0,0), \end{aligned} \tag{6.79}$$

and hence, it follows from Theorem 6.2 that the zero solution $x(t) \overset{\text{a.s.}}{\equiv} 0$ of the closed-loop inverted pendulum system is globally asymptotically stable in probability with the feedback control and stopper policies

$$\begin{aligned} \phi(x) &= -\tfrac{1}{2}R_2^{-1}V'(x)G^{\mathrm{T}}, \\ \psi(x) &= \tfrac{1}{2}R_3^{-1}V'(x)G^{\mathrm{T}}. \end{aligned}$$

Furthermore, the performance functional (6.76), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2\phi(x) - \psi^{\mathrm{T}}(x)R_3\psi(x) - V'(x)f(x) - \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V''(x)D(x),$$

is optimized in the sense that

$$J(x_0, \phi(x(\cdot)), \psi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} \max_{v(\cdot)\in\mathcal{U}_2} J(x_0, u(\cdot), v(\cdot)), \qquad x_0 \in \mathbb{R}^n.$$

With $x(0) \overset{\text{a.s.}}{=} [\pi, 0]^{\mathrm{T}}$, Figure 6.6 shows the sample average along with the standard deviation of the controlled system state versus time, whereas Figures 6.7 and 6.8 show the sample average along with the standard deviation of the corresponding control and stopper signals versus time for 10 sample paths, respectively. △

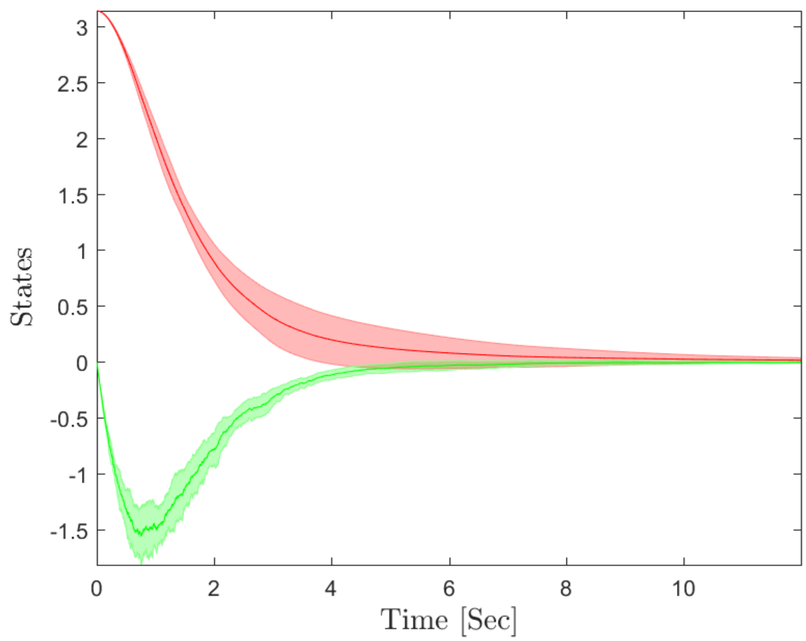

Figure 6.6: Sample average along with the sample standard deviation of the closed-loop system trajectory versus time.

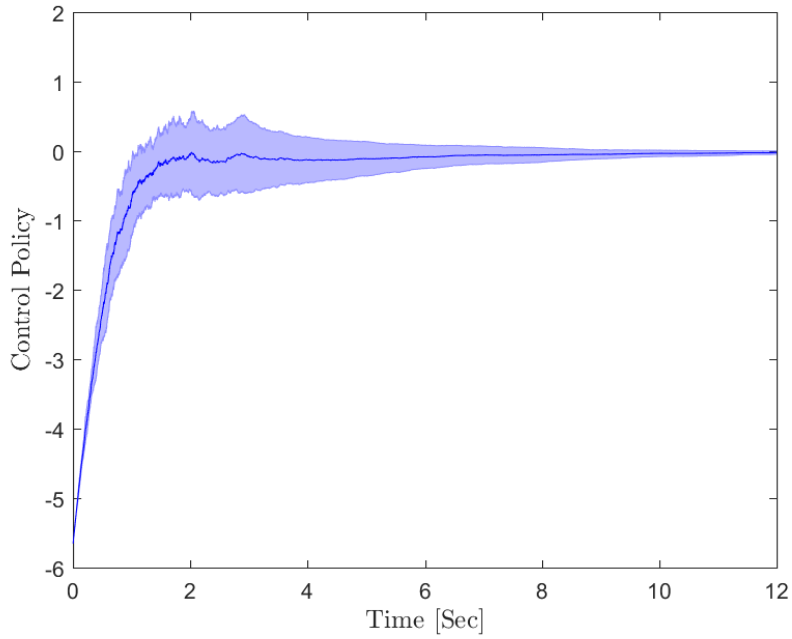

Figure 6.7: Sample average along with the sample standard deviation of the control signal versus time.

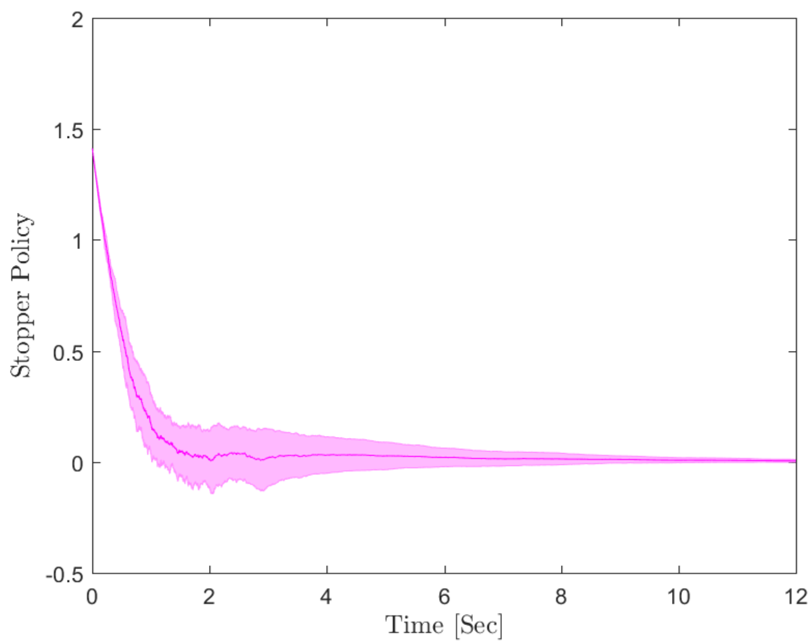

Figure 6.8: Sample average along with the sample standard deviation of the stopper signal versus time.

Chapter 7

Dissipativity Theory for Nonlinear Stochastic Dynamical Systems: Input-Output and State Properties, and Stability of Feedback Interconnections

7.1. Introduction

In this chapter, we develop stochastic dissipativity and losslessness notions for nonlinear stochastic dynamical systems. Specifically, a stochastic version of dissipativity using both an input-output as well as a state dissipation inequality in expectation for controlled Markov diffusion processes is presented. Furthermore, we show that the average stored system energy in a dissipative stochastic dynamical system is a supermartingale with respect to the system filtration and is bounded from below by the mean energy that can be extracted from the system and bounded from above by the mean energy that can be delivered to the stochastic dynamical system in order to transfer it from the origin to an arbitrary nonempty closed or open subset in the state space over a finite stopping time. Moreover, we develop necessary and sufficient extended Kalman-Yakubovich-Popov conditions in terms of the drift and diffusion dynamics for characterizing stochastic dissipativity via two-times continuously differentiable storage functions.

Finally, using the concepts of stochastic dissipativity for stochastic dynamical systems with appropriate storage functions and supply rates, we construct smooth Lyapunov functions for stochastic feedback systems by appropriately combining the storage functions for the forward and feedback subsystems. General stability criteria are given for Lyapunov, asymptotic, and exponential mean square stability in probability for feedback interconnections of stochastic dynamical systems. In the case where the supply rate involves the net system power or weighted input-output energy, these results provide extensions of the classical positivity and small gain theorems to stochastic dynamical systems.

7.2. Stochastic Dissipative and Exponentially Dissipative Dynamical Systems

In this section, we introduce the definition of stochastic dissipativity and stochastic losslessness for general stochastic dynamical systems in terms of an inequality in expectation involving generalized system power input, or supply rate, and a generalized energy function, or storage function. In particular, we consider *open* dynamical systems wherein the system interaction with the environment is explicitly taken into account through the system inputs and system outputs. Specifically, the environment acts on the dynamical system through the system inputs and system disturbance, and the dynamical system reacts through the system outputs.

We begin by considering nonlinear stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = F(x(t),u(t))\mathrm{d}t + D(x(t),u(t))\mathrm{d}w(t), \quad x(t_0) \overset{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{7.1}$$

$$y(t) = H(x(t),u(t)), \tag{7.2}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in \mathcal{H}_m^U$, $U \subseteq \mathbb{R}^m$, $y(t) \in \mathcal{H}_l^Y$, $Y \subseteq \mathbb{R}^l$, $w(\cdot)$ is a d-dimensional independent standard Wiener process and is independent of $x(t_0)$, $F : \mathcal{D} \times U \to \mathbb{R}^n$, $D : \mathcal{D} \times U \to \mathbb{R}^{n\times d}$, and $H : \mathcal{D} \times U \to Y$. For the dynamical system $\mathcal{G}$ given by (7.1) and (7.2) defined on the state space $\mathcal{H}_n^{\mathcal{D}}$, $\mathcal{U}$ and $\mathcal{Y}$ define an input and output space, respectively, consisting of measurable bounded $\mathcal{H}_m^U$-valued and $\mathcal{H}_l^Y$-valued stochastic processes on the semi-infinite interval $[0,\infty)$. The set $\mathcal{H}_m^U$ contains the set of input values with measurable sample paths satisfying a nonanticipativity condition, that is, for every $u(\cdot) \in \mathcal{U}$ and $t \in [0, \infty)$, $u(t) \in \mathcal{H}_m^U$, and for all $t \geq s$, $w(t) - w(s)$ is independent of $u(\tau), w(\tau), \tau \leq s$, and $x(t_0)$. The set $\mathcal{H}_l^Y$ contains the set of output values, that is, for every $y(\cdot) \in \mathcal{Y}$ and $t \in [0, \infty)$, $y(t) \in \mathcal{H}_l^Y$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be closed under the shift operator, that is, if $u(\cdot) \in \mathcal{U}$ (respectively, $y(\cdot) \in \mathcal{Y}$), then the function defined by $u_T \triangleq u(t+T)$ (respectively, $y_T \triangleq y(t+T)$) is contained in $\mathcal{U}$ (respectively, $\mathcal{Y}$) for all $T \geq 0$. We assume that $F(\cdot,\cdot)$, $D(\cdot,\cdot)$, and $H(\cdot,\cdot)$ are continuously differentiable mappings in (x, u) and $\mathcal{G}$ has at least one equilibrium so that, without loss of generality, $F(0,0) = 0$, $D(0,0) = 0$, and $H(0,0) = 0$.

Furthermore, for the nonlinear stochastic dynamical system $\mathcal{G}$ we assume that the required uniform Lipschitz and growth restriction conditions given by (2.4) and (2.5) for the existence and uniqueness of solutions are satisfied, that is, $u(\cdot)$ satisfies sufficient regularity conditions such that the system (7.1) has a unique solution forward in time. For the dynamical system $\mathcal{G}$ given by (7.1) and (7.2), a function $r : U \times Y \to \mathbb{R}$ such that $r(0,0) = 0$ is called a *supply rate* if $r(u(t), y(t))$, is locally Lebesgue integrable for all input-output pairs satisfying (7.1) and (7.2), that is, for all input-output pairs $u(\cdot) \in \mathcal{U}$ and $y(\cdot) \in \mathcal{Y}$ satisfying (7.1) and (7.2), $r(\cdot,\cdot)$ satisfies

$$\mathbb{E}\left[\int_{t_1}^{t_2} |r(u(s), y(s))| \mathrm{d}s\right] < \infty, \quad t_1, t_2 \geq 0.$$

Definition 7.1. A stochastic dynamical system $\mathcal{G}$ of the form (7.1) and (7.2) is *stochastically dissipative with respect to the supply rate* $r(u, y)$ if the *dissipation inequality*

$$0 \leq \mathbb{E}\left[\mathbb{E}\left[\int_{t_0}^{\tau} r(u(s), y(s)) \mathrm{d}s | x(t_0) \overset{\text{a.s.}}{=} 0\right]\right] \tag{7.3}$$

is satisfied for all $\mathcal{F}_t$-stopping times $\tau \overset{\text{a.s.}}{\geq} t_0$ and all $u(\cdot) \in \mathcal{U}$ along the sample paths of $\mathcal{G}$. A stochastic dynamical system $\mathcal{G}$ of the form (7.1) and (7.2) is *stochastically exponentially dissipative with respect to the supply rate* $r(u, y)$ if there exists a constant $\varepsilon > 0$ such that the *stochastic exponential dissipation inequality*

$$0 \leq \mathbb{E}\left[\mathbb{E}\left[\int_{t_0}^{\tau} e^{\varepsilon s} r(u(s), y(s))\mathrm{d}s | x(t_0) = 0\right]\right] \tag{7.4}$$

is satisfied for all $\mathcal{F}_t$-stopping times $\tau \overset{\text{a.s.}}{\geq} t_0$ and all $u(\cdot) \in \mathcal{U}$ along the sample paths of $\mathcal{G}$. A stochastic dynamical system $\mathcal{G}$ of the form (7.1) and (7.2) is *lossless with respect to the supply rate* $r(u, y)$ if $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y)$ and the dissipation inequality (7.3) is satisfied as an equality for all $t \geq t_0$ and all $u(\cdot) \in \mathcal{U}$ with $x(t_0) \overset{\text{a.s.}}{=} x(t) \overset{\text{a.s.}}{=} 0$ along the sample paths of $\mathcal{G}$.

In the following we shall use either 0 or t_0 to denote the initial time for $\mathcal{G}$. Next, define the *available storage* $V_{\mathrm{a}}(x_0)$ of the nonlinear stochastic dynamical system $\mathcal{G}$ by

$$\begin{aligned} V_{\mathrm{a}}(x_0) &\triangleq - \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right] \\ &= \sup_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[-\int_0^{\tau} r(u(t), y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right], \end{aligned} \tag{7.5}$$

where $x(t)$, $t \geq 0$, is the solution to (7.1) with $x(0) \overset{\text{a.s.}}{=} x_0$ and admissible input $u(\cdot) \in \mathcal{U}$. The supremum in (7.5) is taken over all $\mathcal{F}_t$-measurable inputs $u(\cdot)$, all the finite $\mathcal{F}_t$-stopping times $\tau \overset{\text{a.s.}}{\geq} 0$, and all system sample paths with initial value $x(0) \overset{\text{a.s.}}{=} x_0$ and terminal value left free. Note that $V_{\mathrm{a}}(x) \geq 0$ for all $x \in \mathcal{D}$ since $V_{\mathrm{a}}(x)$ is the supremum over a set of numbers containing the zero element ($\tau \overset{\text{a.s.}}{=} 0$). When the final state is not free but rather constrained to $x(t) \overset{\text{a.s.}}{=} 0$ corresponding to the equilibrium of the uncontrolled system, then $V_{\mathrm{a}}(x_0)$ corresponds to the *virtual available storage*. The available storage of a nonlinear stochastic dynamical system $\mathcal{G}$ is the maximum amount of average storage, or generalized average stored energy, which can be extracted from the nonlinear stochastic dynamical system $\mathcal{G}$ at any finite stopping time τ.

Similarly, define the *available exponential storage* $V_{\mathrm{a}}(x_0)$ of the nonlinear dynamical system $\mathcal{G}$ by

$$V_{\mathrm{a}}(x_0) \triangleq - \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} e^{\varepsilon t} r(u(t), y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right], \tag{7.6}$$

where $x(t)$, $t \geq 0$, is the solution to (7.1) with $x(0) \overset{\text{a.s.}}{=} x_0$ and admissible input $u(\cdot) \in \mathcal{U}$. Note that if we define the available exponential storage as the time-varying function

$$\hat{V}_{\text{a}}(x_0, t_0) = -\inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} t_0} \mathbb{E}\left[\mathbb{E}\left[\int_{t_0}^{\tau} e^{\varepsilon t} r(u(t), y(t))\text{d}t | x(t_0) \overset{\text{a.s.}}{=} x_0\right]\right], \tag{7.7}$$

where $x(t)$, $t \geq t_0$, is the solution to (7.1) with $x(t_0) \overset{\text{a.s.}}{=} x_0$ and admissible input $u(\cdot)$, it follows that, since $\mathcal{G}$ is time invariant,

$$\hat{V}_{\text{a}}(x_0, t_0) = -e^{\varepsilon t_0} \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_{0}^{\tau} e^{\varepsilon t} r(u(t), y(t))\text{d}t\right]\right] = e^{\varepsilon t_0} V_{\text{a}}(x_0). \tag{7.8}$$

Hence, an alternative expression for the available exponential storage function $V_{\text{a}}(x_0)$ is given by

$$V_{\text{a}}(x_0) = -e^{-\varepsilon t_0} \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} t_0} \mathbb{E}\left[\mathbb{E}\left[\int_{t_0}^{\tau} e^{\varepsilon t} r(u(t), y(t))\text{d}t | x(t_0) \overset{\text{a.s.}}{=} x_0\right]\right]. \tag{7.9}$$

$\hat{V}_{\text{a}}(x_0, t_0)$ given by (7.7) defines the available storage function for nonstationary (time-varying) stochastic dynamical systems. As shown above, in the case of stochastic exponentially dissipative systems, $\hat{V}_{\text{a}}(x_0, t_0) = e^{\varepsilon t_0} V_{\text{a}}(x_0)$.

Next, we show that the available storage (resp., available exponential storage) is finite and zero at the origin if and only if $\mathcal{G}$ is stochastically dissipative (resp., stochastically exponentially dissipative). For this result we require three more definitions.

Definition 7.2. A nonlinear stochastic dynamical system $\mathcal{G}$ is *completely stochastically reachable* if, for all $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, there exist a finite random variable $\tau_{\mathcal{B}_\varepsilon(x_0)} \overset{\text{a.s.}}{\geq} 0$, called the *first hitting time*, defined by $\tau_{\mathcal{B}_\varepsilon(x_0)}(\omega) \triangleq \inf\{t \geq 0 : x(t, \omega) \in \mathcal{B}_\varepsilon(x_0)\}$, and a square integrable input $u(t)$ defined on $[0, \tau_{\mathcal{B}_\varepsilon(x_0)}]$ such that the state $x(t)$, $t \geq 0$, can be driven from $x(0) \overset{\text{a.s.}}{=} 0$ to $x(\tau_{\mathcal{B}_\varepsilon(x_0)})$ and $\mathbb{E}[\tau_{x_0}] < \infty$, where $\tau_{x_0} \triangleq \sup_{\varepsilon > 0} \tau_{\mathcal{B}_\varepsilon(x_0)}$ and the supremum is taken pointwise.

Definition 7.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2). A measurable function $V_{\text{s}} : \mathcal{D} \to \mathbb{R}$ is called a *storage function* for $\mathcal{G}$ with a supply rate $r(\cdot,\cdot)$ if $V_{\rm s}(\cdot)$ is bounded from below and $V_{\rm s}(x(t)) - \int_{t_0}^{t} r(u(s), y(s)){\rm d}s$, $t \geq t_0$, is a $\mathcal{F}_t$-supermartingale for all $t_0, t \geq 0$, where $x(t)$, $t \geq t_0$, is the solution of (7.1) with $u(\cdot) \in \mathcal{U}$; or, equivalently,

$$\mathbb{E}\left[V_{\rm s}(x(t))|\mathcal{F}_{t_0}\right] \leq V_{\rm s}(x(t_0)) + \mathbb{E}\left[\int_{t_0}^{t} r(u(s), y(s)){\rm d}s|\mathcal{F}_{t_0}\right], \qquad t > t_0. \qquad (7.10)$$

Remark 7.1. If $V_{\rm s}(\cdot)$ is lower bounded, then we can always shift $V_{\rm s}(\cdot)$ so that, with minor abuse of notation, $V_{\rm s}(x) \geq 0$, $x \in \mathbb{R}^n$, and $V_{\rm s}(0) = 0$. Without loss of generality, in the remainder of the chapter we assume that all storage functions are nonnegative.

Inequality (7.10) is a *dissipation inequality* in expectation and reflects the fact that some of the supplied generalized energy to the open dynamical system $\mathcal{G}$ is stored, and some is dissipated. The dissipated generalized energy is nonnegative and is given by the difference of what is supplied and what is stored. In addition, the amount of generalized stored energy is a function of the state of the dynamical system.

Definition 7.4. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2). A measurable function $V_{\rm s} : \mathcal{D} \to \mathbb{R}$ is called an *exponential storage function* for $\mathcal{G}$ with a supply rate $r(\cdot,\cdot)$ if $V_{\rm s}(\cdot)$ is bounded from below and $e^{\varepsilon t}V_{\rm s}(x(t)) - \int_{t_0}^{t} e^{\varepsilon s} r(u(s), y(s)){\rm d}s$, $t \geq t_0$, is a $\mathcal{F}_t$-supermartingale for all $t_0, t \geq 0$, where $x(t)$, $t \geq t_0$, is the solution of (7.1) with $u(\cdot) \in \mathcal{U}$; or, equivalently,

$$\mathbb{E}\left[e^{\varepsilon t}V_{\rm s}(x(t))|\mathcal{F}_{t_0}\right] \leq e^{\varepsilon t_0}V_{\rm s}(x(t_0)) + \mathbb{E}\left[\int_{t_0}^{t} e^{\varepsilon s} r(u(s), y(s)){\rm d}s|\mathcal{F}_{t_0}\right], \qquad t > t_0. \qquad (7.11)$$

Theorem 7.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2), and assume that $\mathcal{G}$ is completely stochastically reachable. Then $\mathcal{G}$ is stochastically dissipative (resp., stochastically exponentially dissipative) with respect to the supply rate $r(u, y)$ if and only if the available system storage $V_{\rm a}(x_0)$ given by (7.5) (resp., the available exponential storage $V_{\rm a}(x_0)$ given by (7.6)) is finite for all $x_0 \in \mathcal{D}$ and $V_{\rm a}(0) = 0$. Moreover, if

$V_{\mathrm{a}}(0) = 0$ and $V_{\mathrm{a}}(x_0)$ is finite for all $x_0 \in \mathcal{D}$, then $V_{\mathrm{a}}(x)$, $x \in \mathcal{D}$, is a storage function (resp., exponential storage function) for $\mathcal{G}$. Finally, all nonnegative definite storage functions (resp., exponential storage functions) $V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, for $\mathcal{G}$ satisfy

$$0 \le V_{\mathrm{a}}(x) \le V_{\mathrm{s}}(x), \quad x \in \mathcal{D}. \tag{7.12}$$

Proof: Suppose $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$. Since $\mathcal{G}$ is completely reachable it follows that for every $x_0 \in \mathcal{D}$ and $\varepsilon > 0$, there exist a finite first hitting time $\tau_{\mathcal{B}_\varepsilon(x_0)} \overset{\text{a.s.}}{\ge} 0$ and an admissible input $\hat{u}(\cdot) \in \mathcal{U}$ defined on $[0, \tau_{\mathcal{B}_\varepsilon(x_0)}]$ such that $x(0) \overset{\text{a.s.}}{=} 0$ and $\mathbb{P}\left(x(\tau_{\mathcal{B}_\varepsilon(x_0)}) \in \mathcal{B}_\varepsilon(x_0)\right) = 1$. Now, since $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$ and $x(0) \overset{\text{a.s.}}{=} 0$ it follows that

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^\tau r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0\right]\right] \ge 0$$

for all $u(\cdot) \in \mathcal{U}$ and all stopping times $\tau \overset{\text{a.s.}}{\ge} 0$, or, equivalently,

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0 + \int_{\tau_{x_0}}^\tau r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0\right]\right] \ge 0, \ \tau \overset{\text{a.s.}}{\ge} \tau_{x_0} \overset{\text{a.s.}}{\ge} 0.$$

Therefore,

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0, u = \hat{u}\right]\right] + \mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^\tau r(u(s), y(s))\mathrm{d}s | \mathcal{F}_{\tau_{x_0}}\right]\right] \ge 0,$$
$$\tau \overset{\text{a.s.}}{\ge} \tau_{x_0} \overset{\text{a.s.}}{\ge} 0,$$

and hence,

$$\mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^\tau r(u(s), y(s))\mathrm{d}s | \mathcal{F}_{\tau_{x_0}}\right]\right] \ge -\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0, u = \hat{u}\right]\right],$$
$$\tau \overset{\text{a.s.}}{\ge} \tau_{x_0} \overset{\text{a.s.}}{\ge} 0,$$

which implies that there exists a function $W : \mathcal{D} \to \mathbb{R}$ such that

$$\mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^\tau r(u(s), y(s))\mathrm{d}s | \mathcal{F}_{\tau_{x_0}}\right]\right] \ge W(x_0) > -\infty, \qquad \tau \overset{\text{a.s.}}{\ge} \tau_{x_0} \overset{\text{a.s.}}{\ge} 0. \tag{7.13}$$

Next, it follows from (7.13) and the strong Markov property of solutions [74] that, for all $x \in \mathcal{D}$,

$$\begin{aligned} V_{\mathrm{a}}(x) &= -\inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right] \\ &= -\inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^{\tau+\tau_{x_0}} r(u(t), y(t))\mathrm{d}t | x(\tau_{x_0}) \overset{\text{a.s.}}{=} x_0\right]\right] \\ &= -\inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^{\tau+\tau_{x_0}} r(u(t), y(t))\mathrm{d}t | \mathcal{F}_{\tau_{x_0}}\right]\right] \\ &\leq -W(x), \end{aligned} \tag{7.14}$$

and hence, the available storage $V_{\mathrm{a}}(x) < \infty$, $x \in \mathcal{D}$. Furthermore, with $x(0) \overset{\text{a.s.}}{=} 0$, it follows that for all admissible inputs $u(t)$, $t \geq 0$,

$$\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0\right]\right] \geq 0, \qquad \tau \overset{\text{a.s.}}{\geq} 0, \tag{7.15}$$

which implies that

$$\sup_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \left[-\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(s), y(s))\mathrm{d}s | x(0) \overset{\text{a.s.}}{=} 0\right]\right]\right] \leq 0, \tag{7.16}$$

or, equivalently, $V_{\mathrm{a}}(0) \leq 0$. However, since $V_{\mathrm{a}}(x) \geq 0$, $x \in \mathcal{D}$, it follows that $V_{\mathrm{a}}(0) = 0$.

Conversely, suppose $V_{\mathrm{a}}(0) = 0$ and $V_{\mathrm{a}}(x_0)$, $x_0 \in \mathcal{D}$, is finite. Now, it follows from (7.5) (with $\tau \overset{\text{a.s.}}{=} 0$) that $V_{\mathrm{a}}(x_0) \geq 0$, $x_0 \in \mathcal{D}$. Next, let $x(t)$, $t \geq 0$, satisfy (7.1) with admissible input $u(t)$, $t \in [t_0, T]$. Since $-V_{\mathrm{a}}(x_0)$, $x_0 \in \mathcal{D}$, is given by the infimum over all admissible inputs $u(\cdot)$ in (7.5), it follows that for all admissible inputs $u(\cdot) \in \mathcal{U}$ and $\mathcal{F}_t$-stopping times $\tau \overset{\text{a.s.}}{\geq} 0$,

$$-V_{\mathrm{a}}(x_0) \leq \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right],$$

which, since by assumption $V_{\mathrm{a}}(0) = 0$, further implies

$$0 \leq \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right]\right].$$

Hence, $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$.

To prove that $V_{\rm a}(x)$ given by (7.5) is a storage function let $\tau \overset{\rm a.s.}{\geq} 0$ be the $\mathcal{F}_t$-stopping time, $x \in \mathcal{D}$, and $t \geq 0$, and note that since $V(x) < \infty$ by assumption,

$$\begin{aligned}
V_{\rm a}(x) &= \sup_{u(\cdot),\,\tau \overset{\rm a.s.}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[-\int_0^\tau r(u(s),y(s)){\rm d}s | x(0) \overset{\rm a.s.}{=} x\right]\right] \\
&\geq \sup_{u(\cdot),\,\tau \overset{\rm a.s.}{\geq} t} \mathbb{E}\left[\mathbb{E}\left[-\int_0^\tau r(u(s),y(s)){\rm d}s | x(0) \overset{\rm a.s.}{=} x\right]\right] \\
&= \sup_{u([0,t])} \mathbb{E}\Bigg[\mathbb{E}\left[-\int_0^t r(u(s),y(s)){\rm d}s | x(0) \overset{\rm a.s.}{=} x\right] \\
&\qquad + \sup_{u(t+\cdot),\,\tau \overset{\rm a.s.}{\geq} t} \mathbb{E}\left[\mathbb{E}\left[-\int_t^\tau r(u(s),y(s)){\rm d}s | x(t)\right] | x(0) \overset{\rm a.s.}{=} x\right]\Bigg] \\
&= \sup_{u([0,t])} \mathbb{E}\left[\mathbb{E}\left[-\int_0^t r(u(s),y(s)){\rm d}s | x(0) \overset{\rm a.s.}{=} x + V_{\rm a}(x(t)) | x(0) \overset{\rm a.s.}{=} x\right]\right].
\end{aligned} \tag{7.17}$$

Now, let $\hat{\tau}, \tau$ be finite $\mathcal{F}_t$-stopping times such that $0 \overset{\rm a.s.}{\leq} \hat{\tau} \overset{\rm a.s.}{\leq} \tau$. Then, using the strong Markov property of solutions to (7.1) [74], it follows from (7.17) that

$$\begin{aligned}
&\mathbb{E}\left[-\int_0^\tau r(u(s),y(s)){\rm d}s | \mathcal{F}_{\hat{\tau}} + V_{\rm a}(x(\tau)) | \mathcal{F}_{\hat{\tau}}\right] \\
&\qquad = -\int_0^{\hat{\tau}} r(u(s),y(s)){\rm d}s + \mathbb{E}\left[-\int_{\hat{\tau}}^\tau r(u(s),y(s)){\rm d}s | \mathcal{F}_{\hat{\tau}} + V_{\rm a}(x(\tau)) | \mathcal{F}_{\hat{\tau}}\right] \\
&\qquad \leq -\int_0^{\hat{\tau}} r(u(s),y(s)){\rm d}s + \mathbb{E}\left[V_{\rm a}(x(\hat{\tau})) | \mathcal{F}_{\hat{\tau}}\right] \\
&\qquad = -\int_0^{\hat{\tau}} r(u(s),y(s)){\rm d}s + V_{\rm a}(x(\hat{\tau})).
\end{aligned} \tag{7.18}$$

Therefore, $V_{\rm a}(x(t)) - \int_0^t r(u(s),y(s)){\rm d}s$, $t \geq 0$, is a $\mathcal{F}_t$-supermartingale, and hence, $V_{\rm a}(\cdot)$ is a storage function for $\mathcal{G}$.

Next, if $V_{\rm s}(x)$, $x \in \mathcal{D}$, is a nonnegative-definite storage function, then it follows from Doob's optional-stopping theorem [111, Thm. 10.10] that, for all $\mathcal{F}_t$-stopping times $\tau \overset{\rm a.s.}{\geq} 0$ and $x_0 \in \mathcal{D}$,

$$\begin{aligned}
V_{\rm s}(x(0)) &\geq \mathbb{E}\left[-\int_0^\tau r(u(t),y(t)){\rm d}t + V_{\rm s}(x(\tau)) | x(0)\right] \\
&\geq \mathbb{E}\left[-\int_0^\tau r(u(t),y(t)){\rm d}t | x(0)\right],
\end{aligned}$$

and hence, with $x(0) \overset{\text{a.s.}}{=} x_0$, it follows that

$$V_{\rm s}(x_0) = \mathbb{E}[V_{\rm s}(x(0))] \geq -\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t)){\rm d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right],$$

which further implies

$$V_{\rm s}(x_0) \geq -\inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t)){\rm d}t | x(0) \overset{\text{a.s.}}{=} x_0\right]\right] = V_{\rm a}(x_0),$$

yielding (7.12).

Finally, the proof for the stochastic exponentially dissipative case follows an identical construction and, hence, is omitted. □

The following corollary to Theorem 7.1 shows that the nonlinear stochastic dynamical system $\mathcal{G}$ is stochastically dissipative (resp., stochastically exponentially dissipative) with respect to the supply rate $r(\cdot,\cdot)$ if and only if there exists a storage function (resp., exponential storage function) $V_{\rm s}(\cdot)$ satisfying (7.10) (resp., (7.11)).

Corollary 7.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2), and assume that $\mathcal{G}$ is completely stochastically reachable. Then $\mathcal{G}$ is stochastically dissipative (resp., stochastically exponentially dissipative) with respect to the supply rate $r(u, y)$ if and only if there exists a nonnegative measurable function $V_{\rm s} : \mathcal{D} \to \mathbb{R}$ satisfying $V_{\rm s}(0) = 0$ and $V_{\rm s}(x(t)) - \int_{t_0}^{t} r(u(s), y(s)){\rm d}s$, $t \geq t_0$ (resp., $e^{\varepsilon t}V_{\rm s}(x(t)) - \int_{t_0}^{t} e^{\varepsilon s} r(u(s), y(s)){\rm d}s$, $t \geq t_0$), is a $\mathcal{F}_t$-supermartingale for all $t_0, t \geq 0$, where $x(t)$, $t \geq t_0$, is the solution of (7.1) with $u(\cdot) \in \mathcal{U}$.

Proof: The result is immediate from Theorem 7.1 with $V_{\rm s}(x) = V_{\rm a}(x)$. □

The following theorem provides conditions for guaranteeing that all storage functions (resp., exponential storage functions) of a given stochastically dissipative (resp., stochastically exponentially dissipative) nonlinear stochastic dynamical system are positive definite. For this result we require the following definition.

Definition 7.5. A nonlinear stochastic dynamical system $\mathcal{G}$ is *zero-state observable* if $u(t) \stackrel{\text{a.s.}}{\equiv} 0$ and $y(t) \stackrel{\text{a.s.}}{\equiv} 0$ implies $x(t) \stackrel{\text{a.s.}}{\equiv} 0$.

Theorem 7.2. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2), and assume that $\mathcal{G}$ is completely stochastically reachable and zero-state observable. Furthermore, assume that $\mathcal{G}$ is stochastically dissipative (resp., stochastically exponentially dissipative) with respect to the supply rate $r(u, y)$ and there exists a function $\kappa : Y \to U$ such that $\kappa(0) = 0$ and $r(\kappa(y), y) < 0$, $y \neq 0$. Then all the storage functions (resp., exponential storage functions) $V_{\rm s}(x)$, $x \in \mathcal{D}$, for $\mathcal{G}$ are positive definite, that is, $V_{\rm s}(0) = 0$ and $V_{\rm s}(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$.

Proof: It follows from Theorem 7.1 that the available storage $V_{\rm a}(x)$, $x \in \mathcal{D}$, is a storage function for $\mathcal{G}$. Next, suppose there exists $x \in \mathcal{D}$, $x \neq 0$, such that $V_{\rm a}(x) = 0$, which implies that

$$\begin{aligned} 0 &= \sup_{u(\cdot),\, \tau \stackrel{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[-\int_0^\tau r(u(t), y(t))\mathrm{d}t | x(0) \stackrel{\text{a.s.}}{=} x\right]\right] \\ &\geq \sup_{\tau \stackrel{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[-\int_0^\tau r(\kappa(y(t)), y(t))\mathrm{d}t | x(0) \stackrel{\text{a.s.}}{=} x\right]\right] \\ &\geq 0, \end{aligned}$$

and, hence, $r(\kappa(y(t)), y(t)) \stackrel{\text{a.s.}}{=} 0$ almost everywhere $t \geq 0$. Since there exists a function $\kappa : Y \to U$ such that $\kappa(0) = 0$ and $r(\kappa(y), y) < 0$, $y \neq 0$, it follows that $y(t) \stackrel{\text{a.s.}}{=} 0$ almost everywhere $t \geq 0$. Now, since $\mathcal{G}$ is zero-state observable it follows that $x = 0$, and hence, $V_{\rm a}(x) = 0$ if and only if $x = 0$. The result now follows from (7.12). Finally, the proof for the exponentially dissipative case is identical. □

If $V_{\rm s}(\cdot)$ is two-times continuously differentiable, then an equivalent statement for the stochastic dissipativeness of $\mathcal{G}$ with respect to the supply rate $r(u, y)$ can be characterized by the infinitesimal generator $\mathcal{L}$.

Proposition 7.1. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2). If $V_{\rm s} : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ is two-times continuously differentiable and has a compact support, then $\mathcal{G}$ is stochastically dissipative with respect to supply rate $r(\cdot,\cdot)$ if and only if

$$\mathcal{L}V_{\rm s}(x) \triangleq \frac{\partial V(x)}{\partial x}F(x,u) + \frac{1}{2}{\rm tr}\ D^{\rm T}(x)\frac{\partial^2 V(x)}{\partial x^2}D(x) \leq r(u, H(x,u)), \quad (x,u) \in \mathcal{D}\times U. \quad (7.19)$$

Proof: Suppose $\mathcal{G}$ is dissipative with respect to the supply rate $r(u,y)$ and with a storage function $V_{\rm s}(x)$, $x \in \mathcal{D}$. Then, for every $h > 0$, it follows from Definition 7.3 that

$$\mathbb{E}\left[V_{\rm s}(x(t+h))|\mathcal{F}_t - \int_{t_0}^{t+h} r(u(s),y(s)){\rm d}s|\mathcal{F}_t\right] \leq V_{\rm s}(x(t)) - \int_{t_0}^{t} r(u(s),y(s)){\rm d}s, \qquad (7.20)$$

which implies

$$\begin{aligned}\mathbb{E}[V_{\rm s}(x(t+h))|\mathcal{F}_t] - V_{\rm s}(x(t)) &\leq \mathbb{E}\left[\int_{t_0}^{t+h} r(u(s),y(s)){\rm d}s|\mathcal{F}_t\right] - \int_{t_0}^{t} r(u(s),y(s)){\rm d}s \\ &= \mathbb{E}\left[\int_{t}^{t+h} r(u(s),y(s)){\rm d}s|\mathcal{F}_t\right]. \qquad (7.21)\end{aligned}$$

Using the Markov property of solutions, taking the expectation, and letting $x(t) \overset{\rm a.s.}{=} x$, $x \in \mathcal{D}$, and $u(t) \overset{\rm a.s.}{=} u$, $u \in U$, yields

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}\left[V_{\rm s}(x(t+h))|x(t) \overset{\rm a.s.}{=} x, u(t) \overset{\rm a.s.}{=} u\right]\right] - V_{\rm s}(x) \\ &\qquad \leq \mathbb{E}\left[\mathbb{E}\left[\int_{t}^{t+h} r(u(s),H(x(s),u(s))){\rm d}s|x(t) \overset{\rm a.s.}{=} x, u(t) \overset{\rm a.s.}{=} u\right]\right].\end{aligned}$$

Now, dividing both sides by h and taking the limit as $h \to 0^+$ yields

$$\begin{aligned}&\lim_{h\to 0^+}\frac{1}{h}\left[\mathbb{E}\left[\mathbb{E}\left[V_{\rm s}(x(t+h))|x(t) \overset{\rm a.s.}{=} x, u(t) \overset{\rm a.s.}{=} u\right]\right] - V_{\rm s}(x)\right] \\ &\qquad \leq \lim_{h\to 0^+}\frac{1}{h}\left[\mathbb{E}\left[\mathbb{E}\left[\int_{t}^{t+h} r(u(s),H(x(s),u(s))){\rm d}s|x(t) \overset{\rm a.s.}{=} x, u(t) \overset{\rm a.s.}{=} u\right]\right]\right] \\ &\qquad = r(u, H(x,u)).\end{aligned}$$

Since $V_{\rm s} \in {\rm C}^2$ and has a compact support by assumption, (7.19) follows from [83, Thm. 7.9].

Conversely, if $V_{\rm s} \in {\rm C}^2$ and has a compact support, and satisfies (7.19), then the infinitesimal generator operator $\mathcal{L}$ of the process $V(x(t))$, $t > t_0$, where $x(t)$, $t > t_0$, is solution of

(7.1), is given by (7.19) [83, Thm. 7.9]. Now, letting $0 \leq t_1 \leq t_2$,

$$\begin{aligned}
&\mathbb{E}\left[V_{\mathrm{s}}(x(t_2))|\mathcal{F}_{t_1} - \int_0^{t_2} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{t_1}\right] - \mathbb{E}\left[V_{\mathrm{s}}(x(t_1))|\mathcal{F}_{t_1} - \int_0^{t_1} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{t_1}\right] \\
&\quad = \mathbb{E}\left[(V_{\mathrm{s}}(x(t_2)) - V_{\mathrm{s}}(x(t_1)))\,|\mathcal{F}_{t_1}\right] - \mathbb{E}\left[\int_{t_1}^{t_2} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{t_1}\right] \\
&\quad = \mathbb{E}\left[\int_{t_1}^{t_2} \mathrm{d}V_{\mathrm{s}}(t)|\mathcal{F}_{t_1}\right] - \mathbb{E}\left[\int_{t_1}^{t_2} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{t_1}\right] \\
&\quad = \mathbb{E}\left[\int_{t_1}^{t_2} \mathcal{L}V_{\mathrm{s}}(t)|\mathcal{F}_{t_1}\right] - \mathbb{E}\left[\int_{t_1}^{t_2} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{t_1}\right] \\
&\quad \leq 0,
\end{aligned} \tag{7.22}$$

which shows that $V_{\mathrm{s}}(x(t)) - \int_0^t r(u(s),y(s))\mathrm{d}s$, $t \geq 0$, is $\mathcal{F}_t$-supermartingale, and hence, $V_{\mathrm{s}}(\cdot)$ is a storage function, and, by definition, $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u,y)$. □

Similarly, an equivalent statement for stochastic exponential dissipativeness of $\mathcal{G}$ with respect to the supply rate $r(u,y)$ is

$$\mathcal{L}V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) \leq r(u, H(x,u)). \tag{7.23}$$

Next, we introduce the concept of a required supply of a nonlinear stochastic dynamical system. Specifically, define the *required supply* $V_{\mathrm{r}}(x_0)$ of the nonlinear stochastic dynamical system $\mathcal{G}$ by

$$V_{\mathrm{r}}(x_0) = \inf_{u(\cdot),\, \tau_{x_0} \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(t),y(t))\mathrm{d}t|x(0) \overset{\text{a.s.}}{=} 0\right]\right], \tag{7.24}$$

where $x(t)$, $t \geq 0$, is the solution to (7.1). The infimum in (7.24) is taken over all system sample paths starting from $x(0) \overset{\text{a.s.}}{=} 0$ at time $t = 0$ and ending at $x(\tau_{x_0}) \overset{\text{a.s.}}{=} x_0$ at time $t = \tau_{x_0}$, and all times $t \geq 0$ or, equivalently, over all admissible inputs $u(\cdot)$ which drive the dynamical system $\mathcal{G}$ from the origin to x_0 over the time interval $[0, \tau_{x_0}]$. If the system is not reachable from the origin, then we define $V_{\mathrm{r}}(x_0) = \infty$. It follows from (7.24) that the required supply of a nonlinear stochastic dynamical system is the minimum amount of generalized average energy that has to be delivered to the dynamical system in order to transfer it from an initial state $x(0) \stackrel{\text{a.s.}}{=} 0$ to a given state $x(\tau_{x_0}) \stackrel{\text{a.s.}}{=} x_0$. Similarly, define the *required exponential supply* of the nonlinear dynamical system $\mathcal{G}$ by

$$V_{\rm r}(x_0) = \inf_{u(\cdot),\, \tau_{x_0} \stackrel{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} e^{\varepsilon t} r(u(t), y(t)) \mathrm{d}t | x(0) \stackrel{\text{a.s.}}{=} 0\right]\right], \tag{7.25}$$

where $x(t)$, $t \geq 0$, is the solution to (7.1) with $x(0) \stackrel{\text{a.s.}}{=} 0$ and $x(\tau_{x_0}) \stackrel{\text{a.s.}}{=} x_0$. Note that since, with $x(0) \stackrel{\text{a.s.}}{=} 0$, the infimum in (7.24) is zero it follows that $V_{\rm r}(0) = 0$.

Next, using the notion of a required supply, we show that all storage functions are bounded from above by the required supply and bounded from below by the available storage, and hence, a stochastic dissipative dynamical system can deliver to its surroundings only a fraction of its generalized stored energy and can store only a fraction of the generalized work done to it.

Theorem 7.3. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2), and assume that $\mathcal{G}$ is completely stochastically reachable. Then $\mathcal{G}$ is stochastically dissipative (resp., stochastically exponentially dissipative) with respect to the supply rate $r(u, y)$ if and only if $0 \leq V_{\rm r}(x) < \infty$, $x \in \mathcal{D}$. Moreover, if $V_{\rm r}(x)$ is finite and nonnegative for all $x \in \mathcal{D}$, then $V_{\rm r}(x)$, $x \in \mathcal{D}$, is a storage function (resp., exponential storage function) for $\mathcal{G}$. Finally, all nonnegative storage functions (resp., exponential storage functions) $V_{\rm s}(x)$, $x \in \mathcal{D}$, for $\mathcal{G}$ satisfy

$$0 \leq V_{\rm a}(x) \leq V_{\rm s}(x) \leq V_{\rm r}(x) < \infty, \qquad x \in \mathcal{D}. \tag{7.26}$$

Proof: It follows from the definition of $V_{\rm r}(\cdot)$ that for every stopping time $\tau \stackrel{\text{a.s.}}{\geq} 0$,

$$V_{\rm r}(x(\tau)) = \inf_{u(\cdot),\, \tau \stackrel{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\int_0^{\tau} r(u(t), y(t)) \mathrm{d}t | x(0) \stackrel{\text{a.s.}}{=} 0\right]. \tag{7.27}$$

Now, suppose $\mathcal{G}$ is dissipative with respect to the supply rate $r(u, y)$ then, by definition,

$$0 \leq \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t), y(t)) \mathrm{d}t | x(0) \stackrel{\text{a.s.}}{=} 0\right]\right] < \infty$$

for all $\mathcal{F}_t$-stopping times $\tau \overset{\text{a.s.}}{\geq} 0$ and admissible inputs $u(\cdot)$. Therefore,

$$0 \leq \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^\tau r(u(t),y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right]\right] = \mathbb{E}[V_{\mathrm{r}}(x(\tau))] < \infty,$$

which, with $\tau = \tau_x$, $x \in \mathcal{D}$, implies $V_{\mathrm{r}}(x) = \mathbb{E}[V_{\mathrm{r}}(x(\tau_x))]$, and hence, $0 \leq V_{\mathrm{r}}(x) < \infty$.

Conversely, suppose $0 \leq V_{\mathrm{r}}(x) < \infty$, $x \in \mathcal{D}$. Therefore, for all $\tau \overset{\text{a.s.}}{\geq} 0$,

$$\begin{aligned} 0 &\leq \mathbb{E}[V_{\mathrm{r}}(x(\tau))] \\ &= \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^\tau r(u(t),y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right]\right] \\ &\leq \mathbb{E}\left[\mathbb{E}\left[\int_0^\tau r(u(t),y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right]\right]. \end{aligned}$$

Hence, $\mathcal{G}$ is dissipative with respect to the supply rate $r(u,y)$. To prove that $V_{\mathrm{r}}(\cdot)$ given by (7.24) is a storage function, let $\hat{\tau}, \tau$ be finite $\mathcal{F}_t$-stopping times such that $0 \overset{\text{a.s.}}{\leq} \hat{\tau} \overset{\text{a.s.}}{\leq} \tau$. Then, it follows from (7.27) that

$$\begin{aligned} V_{\mathrm{r}}(x(\tau)) &= \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\int_0^\tau r(u(t),y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right] \\ &\leq \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} \hat{\tau} \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\int_0^\tau r(u(t),y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right] \\ &\leq \inf_{u(\cdot),\, \hat{\tau} \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\int_0^{\hat{\tau}} r(u(t),y(t))\mathrm{d}t | x(0) \overset{\text{a.s.}}{=} 0\right] + \mathbb{E}\left[\int_{\hat{\tau}}^\tau r(u(t),y(t))\mathrm{d}t | x(\hat{\tau})\right] \\ &= V_{\mathrm{r}}(x(\hat{\tau})) + \mathbb{E}\left[\int_{\hat{\tau}}^\tau r(u(t),y(t))\mathrm{d}t | x(\hat{\tau})\right]. \end{aligned} \tag{7.28}$$

Using the strong Markov property of the solution $x(t), t \geq 0$ of (7.1) [74], it follows from (7.28) that

$$\begin{aligned} &\mathbb{E}\left[-\int_0^\tau r(u(s),y(s))\mathrm{d}s | \mathcal{F}_{\hat{\tau}} + V_{\mathrm{r}}(x(\tau)) | \mathcal{F}_{\hat{\tau}}\right] \\ &\quad = \mathbb{E}\left[-\int_0^{\hat{\tau}} r(u(s),y(s))\mathrm{d}s | \mathcal{F}_{\hat{\tau}} - \int_{\hat{\tau}}^\tau r(u(s),y(s))\mathrm{d}s | \mathcal{F}_{\hat{\tau}}\right] + \mathbb{E}\left[V_{\mathrm{r}}(x(\tau)) | \mathcal{F}_{\hat{\tau}}\right] \\ &\quad = \mathbb{E}\left[-\int_0^{\hat{\tau}} r(u(s),y(s))\mathrm{d}s | \mathcal{F}_{\hat{\tau}}\right] + \mathbb{E}\left[V_{\mathrm{r}}(x(\tau)) | \mathcal{F}_{\hat{\tau}} - \int_{\hat{\tau}}^\tau r(u(s),y(s))\mathrm{d}s | \mathcal{F}_{\hat{\tau}}\right] \\ &\quad \leq -\int_0^{\hat{\tau}} r(u(s),y(s))\mathrm{d}s + V_{\mathrm{r}}(x(\hat{\tau})). \end{aligned} \tag{7.29}$$

Therefore, $V_{\rm r}(x(t)) - \int_0^t r(u(s), y(s))\mathrm{d}s$, $t \geq 0$, is $\mathcal{F}_t$-supermartingale, and hence, $V_{\rm r}(\cdot)$ is a storage function for $\mathcal{G}$.

Next, if $V_{\rm s}(x)$, $x \in \mathcal{D}$, is a nonnegative-definite storage function, then it follows from Doob's optional-stopping theorem [111, Thm. 10.10] that, for all $\mathcal{F}_t$-stopping times $\tau \overset{\rm a.s.}{\geq} 0$,

$$V_{\rm s}(x(0)) \geq \mathbb{E}\left[-\int_0^{\tau} r(u(t), y(t))\mathrm{d}t|x(0) + V_{\rm s}(x(\tau))|x(0)\right],$$

Now, let $x(0) \overset{\rm a.s.}{=} 0$ and $\tau = \tau_x, x \in \mathcal{D}$, and since $V_{\rm s}(0) = 0$, it follows that

$$\begin{aligned} 0 &= \mathbb{E}[V_{\rm s}(x(0))] \\ &\geq -\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_x} r(u(t), y(t))\mathrm{d}t|x(0) \overset{\rm a.s.}{=} 0\right]\right] + \mathbb{E}\left[\mathbb{E}\left[V_{\rm s}(x(\tau_x))|x(0) \overset{\rm a.s.}{=} 0\right]\right] \\ &= -\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_x} r(u(t), y(t))\mathrm{d}t|x(0) \overset{\rm a.s.}{=} 0\right]\right] + V_{\rm s}(x), \end{aligned}$$

and hence,

$$V_{\rm s}(x) \leq \inf_{u(\cdot),\, \tau_x \overset{\rm a.s.}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_x} r(u(t), y(t))\mathrm{d}t|x(0) \overset{\rm a.s.}{=} 0\right]\right] = V_{\rm r}(x) < \infty,$$

which implies (7.26).

Finally, the proof for the stochastic exponentially dissipative case follows a similar construction and, hence, is omitted. □

As a direct consequence of Theorems 7.1 and 7.3, we show that the set of all possible storage functions of a stochastic dynamical system forms a convex set parameterized by the system available storage and the system required supply. An identical result holds for exponential storage functions.

Proposition 7.2. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2) with available storage $V_{\rm a}(x)$, $x \in \mathcal{D}$, and required supply $V_{\rm r}(x)$, $x \in \mathcal{D}$, and assume $\mathcal{G}$ is completely stochastically reachable. Then, for every $\alpha \in [0, 1]$,

$$V_{\rm s}(x) = \alpha V_{\rm a}(x) + (1-\alpha)V_{\rm r}(x), \qquad x \in \mathcal{D}, \tag{7.30}$$

is a storage function for $\mathcal{G}$.

Proof: The result is a direct consequence of the definition of storage function by noting that if $V_{\rm a}(x(t)) - \int_0^t r(u(s), y(s)){\rm d}s$, $t \geq 0$, and $V_{\rm r}(x(t)) - \int_0^t r(u(s), y(s)){\rm d}s$, $t \geq 0$, are $\mathcal{F}_t$-supermartingales, then $V_{\rm s}(x(t)) - \int_0^t r(u(s), y(s)){\rm d}s$, $t \geq 0$, is a $\mathcal{F}_t$-supermartingale. □

In light of Theorems 7.1 and 7.3 we have the following result on lossless stochastic dynamical systems.

Theorem 7.4. Consider the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.1) and (7.2), and assume $\mathcal{G}$ is completely stochastically reachable to and from the origin. Then $\mathcal{G}$ is lossless with respect to the supply rate $r(u, y)$ if and only if there exists a storage function $V_{\rm s}(x)$, $x \in \mathcal{D}$, such that $V_{\rm s}(x(t)) - \int_0^t r(u(s), y(s)){\rm d}s$, $t \geq 0$, is a $\mathcal{F}_t$-martingale. Furthermore, if $\mathcal{G}$ is lossless with respect to the supply rate $r(u, y)$, then $V_{\rm a}(x) = V_{\rm r}(x)$, and hence, the storage function $V_{\rm s}(x)$, $x \in \mathcal{D}$, is unique and is given by

$$\begin{aligned} V_{\rm s}(x_0) &= -\mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_0} r(u(t), y(t)){\rm d}t | x(0) \stackrel{\rm a.s.}{=} x_0\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(t), y(t)){\rm d}t | x(0) \stackrel{\rm a.s.}{=} 0\right]\right], \end{aligned} \quad (7.31)$$

where $x(t)$, $t \geq 0$, is the solution to (7.1) with admissible $u(\cdot) \in \mathcal{U}$ and for every $\tau_0, \tau_{x_0} \stackrel{\rm a.s.}{\geq} 0$ such that $x(\tau_0) \stackrel{\rm a.s.}{=} 0$ and $x(\tau_{x_0}) \stackrel{\rm a.s.}{=} x_0$, $x_0 \in \mathcal{D}$.

Proof: Suppose $\mathcal{G}$ is lossless with respect to the supply rate $r(u, y)$. Since $\mathcal{G}$ is completely reachable to and from the origin it follows that, for every $x_0 \in \mathcal{D}$, there exist finite $\tau_{x_0} \stackrel{\rm a.s.}{\geq} 0$ and admissible $u(\cdot) \in \mathcal{U}$ on $[0, \tau_{x_0}]$ such that $x(\tau_{x_0}) \stackrel{\rm a.s.}{=} x_0$ for $x(0) \stackrel{\rm a.s.}{=} 0$. Let $\tau \stackrel{\rm a.s.}{\geq} 0$ be a $\mathcal{F}_t$-stopping time and note that by the strong Markov property of the solution $x(t)$ of (7.1) [74],

$$\begin{aligned} 0 &= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_0+\tau_{x_0}} r(u(t), y(t)){\rm d}t | x(0) \stackrel{\rm a.s.}{=} 0\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\left(\int_0^{\tau_{x_0}} r(u(t), y(t)){\rm d}t + \int_{\tau_{x_0}}^{\tau_0+\tau_{x_0}} r(u(t), y(t)){\rm d}t\right) | x(0) \stackrel{\rm a.s.}{=} 0\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(t), y(t)){\rm d}t | x(0) \stackrel{\rm a.s.}{=} 0\right]\right] + \mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^{\tau_0+\tau_{x_0}} r(u(t), y(t)){\rm d}t | \mathcal{F}_{\tau_{x_0}}\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(t),y(t))\mathrm{d}t|x(0) \overset{\text{a.s.}}{=} 0\right]\right] + \mathbb{E}\left[\mathbb{E}\left[\int_{\tau_{x_0}}^{\tau_0+\tau_{x_0}} r(u(t),y(t))\mathrm{d}t|x(\tau_{x_0}) \overset{\text{a.s.}}{=} x_0\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(t),y(t))\mathrm{d}t|x(0) \overset{\text{a.s.}}{=} 0\right]\right] + \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_0} r(u(t),y(t))\mathrm{d}t|x(0) \overset{\text{a.s.}}{=} x_0\right]\right] \\
&\geq \inf_{u(\cdot),\, \tau_{x_0} \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau_{x_0}} r(u(t),y(t))\mathrm{d}t|x(0) \overset{\text{a.s.}}{=} 0\right]\right] \\
&\quad + \inf_{u(\cdot),\, \tau \overset{\text{a.s.}}{\geq} 0} \mathbb{E}\left[\mathbb{E}\left[\int_0^{\tau} r(u(t),y(t))\mathrm{d}t|x(0) \overset{\text{a.s.}}{=} x_0\right]\right] \\
&= V_{\mathrm{r}}(x_0) - V_{\mathrm{a}}(x_0), \qquad (7.32)
\end{aligned}
$$

which implies that $V_{\mathrm{r}}(x_0) \leq V_{\mathrm{a}}(x_0)$, $x_0 \in \mathcal{D}$. However, since by definition $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u,y)$ it follows from Theorem 7.3 that $V_{\mathrm{a}}(x_0) \leq V_{\mathrm{r}}(x_0)$, $x_0 \in \mathcal{D}$, and hence, every storage function $V_{\mathrm{s}}(x_0)$, $x_0 \in \mathcal{D}$, satisfies $V_{\mathrm{a}}(x_0) = V_{\mathrm{s}}(x_0) = V_{\mathrm{r}}(x_0)$. Furthermore, it follows that the inequality in (7.32) is indeed an equality, which implies (7.31).

Next, let τ_1, τ_2 be two $\mathcal{F}_t$-stopping times such that $\tau_0 \overset{\text{a.s.}}{\geq} \tau_1 \overset{\text{a.s.}}{\geq} \tau_2 \overset{\text{a.s.}}{\geq} 0$, $x(\tau_0) \overset{\text{a.s.}}{=} 0$. It follows from (7.31) that

$$
V_{\mathrm{s}}(x(t)) = -\mathbb{E}\left[\int_t^{\tau_0} r(u(s),y(s))\mathrm{d}s|x(t)\right], \quad \tau_0 \overset{\text{a.s.}}{\geq} t,
$$

and hence,

$$
\begin{aligned}
&\mathbb{E}\left[V_{\mathrm{s}}(x(\tau_1)) - \int_0^{\tau_1} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{\tau_2}\right] \\
&\qquad = \mathbb{E}\left[-\mathbb{E}\left[\int_{\tau_1}^{\tau_0} r(u(s),y(s))\mathrm{d}s|x(\tau_1)\right] - \int_0^{\tau_1} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{\tau_2}\right] \\
&\qquad = \mathbb{E}\left[-\mathbb{E}\left[\int_{\tau_2}^{\tau_0} r(u(s),y(s))\mathrm{d}s|x(\tau_2)\right] - \int_0^{\tau_2} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{\tau_2}\right] \\
&\qquad = \mathbb{E}\left[V_{\mathrm{s}}(x(\tau_2)) - \int_0^{\tau_2} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{\tau_2}\right] \\
&\qquad = V_{\mathrm{s}}(x(\tau_2)) - \int_0^{\tau_2} r(u(s),y(s))\mathrm{d}s,
\end{aligned}
$$

which implies that $V_{\mathrm{s}}(x(t)) - \int_0^t r(u(s),y(s))\mathrm{d}s$, $t \geq 0$, is a $\mathcal{F}_t$-martingale.

Conversely, if there exists a storage function $V_{\mathrm{s}}(x)$, $x \in \mathcal{D}$, such that

$$
V_{\mathrm{s}}(x(t)) - \int_0^t r(u(s),y(s))\mathrm{d}s, \quad t \geq 0,
$$

is a $\mathcal{F}_t$-martingale, then it follows from Corollary 7.1 that $\mathcal{G}$ is stochastically dissipative with respect to the supply rate $r(u, y)$. Furthermore, for every $u(\cdot) \in \mathcal{U}$, $t \geq 0$, and $x(t_0) \stackrel{\text{a.s.}}{=} x(t) \stackrel{\text{a.s.}}{=} 0$, it follows from (7.10) (with an equality) that

$$\int_{t_0}^{t} r(u(s), y(s)) \mathrm{d}s \stackrel{\text{a.s.}}{=} 0,$$

which implies that $\mathcal{G}$ is stochastically lossless with respect to the supply rate $r(u, y)$. □

7.3. Extended Kalman-Yakubovich-Popov Conditions for Nonlinear Stochastic Dynamical Systems

In this section, we show that stochastic dissipativeness, stochastic exponential dissipativeness, and stochastic losslessness of nonlinear affine stochastic dynamical systems $\mathcal{G}$ of the form

$$\mathrm{d}x(t) = [f(x(t)) + G(x(t))u(t)]\mathrm{d}t + D(x(t))\mathrm{d}w(t), \quad x(t_0) \stackrel{\text{a.s.}}{=} x_0, \quad t \geq t_0, \tag{7.33}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{7.34}$$

where for $t \geq t_0$, $x(t) \in \mathcal{H}_n^{\mathcal{D}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n \times m}$, $D : \mathcal{D} \to \mathbb{R}^{n \times d}$, $h : \mathcal{D} \to \mathbb{R}^l$, and $J : \mathcal{D} \to \mathbb{R}^{l \times m}$, can be characterized in terms of the system functions $f(\cdot)$, $G(\cdot)$, $D(\cdot)$, $h(\cdot)$, and $J(\cdot)$. We assume that $f(\cdot)$, $G(\cdot)$, $D(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are continuously differentiable mappings and $\mathcal{G}$ has at least one equilibrium so that, without loss of generality, $f(0) = 0$, $D(0) = 0$, and $h(0) = 0$. Furthermore, for the nonlinear stochastic dynamical system $\mathcal{G}$ we assume that the required properties for the existence and uniqueness of solutions in forward time are satisfied.

For the following result we consider the special case of dissipative systems with quadratic supply rates [108]. Specifically, set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^l$, let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l \times m}$ be given, where $\mathbb{S}^q$ denotes the set of $q \times q$ symmetric matrices, and assume $r(u, y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $r(\kappa(y), y) < 0$, $y \neq 0$, and the available storage $V_{\mathrm{a}}(x)$, $x \in \mathbb{R}^n$, for $\mathcal{G}$ is a two-times continuously differentiable function.

Theorem 7.5. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be zero-state observable and completely stochastically reachable. $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u,y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ if and only if there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\mathrm{s}}(\cdot)$ is two-times continuously differentiable and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\mathrm{s}}'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{7.35}$$

$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{7.36}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x). \tag{7.37}$$

If, alternatively,

$$\mathcal{N}(x) \triangleq R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) > 0, \quad x \in \mathbb{R}^n, \tag{7.38}$$

then $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u,y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ if and only if there exists a two-times continuously differentiable function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 \geq V_{\mathrm{s}}'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''D(x) - h^{\mathrm{T}}(x)Qh(x) + [\tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S)] \cdot\mathcal{N}^{-1}(x)[\tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S)]^{\mathrm{T}}. \tag{7.39}$$

Proof: First, suppose that there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\mathrm{s}}(\cdot)$ is two-times continuously differentiable and positive definite, and (7.35)–(7.37) are satisfied. Then for every admissible input $u(\cdot) \in \mathcal{U}$, $t_1, t_2 \in \mathbb{R}$, $t_2 \geq t_1 \geq t_0$, it follows from (7.35)–(7.37) that

$$\begin{aligned}\mathbb{E}\left[\int_{t_1}^{t_2} r(u,y)\mathrm{d}t|\mathcal{F}_{t_1}\right] &= \mathbb{E}\left[\int_{t_1}^{t_2} \left[y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\right]\mathrm{d}t|\mathcal{F}_{t_1}\right] \\ &= \mathbb{E}\Bigg[\int_{t_1}^{t_2} \big[h^{\mathrm{T}}(x)Qh(x) + 2h^{\mathrm{T}}(x)(S + QJ(x))u \\ &\qquad + u^{\mathrm{T}}(J^{\mathrm{T}}(x)QJ(x) + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + R)u\big]\mathrm{d}t|\mathcal{F}_{t_1}\Bigg]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\bigg[\int_{t_1}^{t_2} \big[V_{\rm s}'(x)(f(x)+G(x)u) + \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x)V_{\rm s}''(x)D(x) + \ell^{\rm T}(x)\ell(x) \\
&\quad +2\ell^{\rm T}(x)\mathcal{W}(x)u + u^{\rm T}\mathcal{W}^{\rm T}(x)\mathcal{W}(x)u\big]{\rm d}t|\mathcal{F}_{t_1}\bigg] \\
&= \mathbb{E}\left[\int_{t_1}^{t_2} \big[\mathcal{L}V_{\rm s}(x) + [\ell(x)+\mathcal{W}(x)u]^{\rm T}[\ell(x)+\mathcal{W}(x)u]\big]{\rm d}t|\mathcal{F}_{t_1}\right] \\
&\geq \mathbb{E}\left[V_{\rm s}(x(t_2))|\mathcal{F}_{t_1}\right] - V_{\rm s}(x(t_1)),
\end{aligned}
$$

where $x(t)$, $t \geq 0$, satisfies (7.33) and $\mathcal{L}V_{\rm s}(\cdot)$ denotes the infinitesimal generator of the storage function along the trajectories $x(t)$, $t \geq t_0$, of (7.33). Now, the result is immediate from Corollary 7.1.

Conversely, suppose that $\mathcal{G}$ is stochastically dissipative with respect to a quadratic supply rate $r(u,y)$. Now, it follows from Theorem 7.1 that the available storage $V_{\rm a}(x)$ of $\mathcal{G}$ is finite for all $x \in \mathbb{R}^n$, $V_{\rm a}(0) = 0$, and

$$
\mathbb{E}\left[V_{\rm a}(x(t_2))|\mathcal{F}_{t_1}\right] \leq V_{\rm a}(x(t_1)) + \mathbb{E}\left[\int_{t_1}^{t_2} r(u(t),y(t)){\rm d}t|\mathcal{F}_{t_1}\right], \quad t_2 \geq t_1, \tag{7.40}
$$

for all admissible $u(\cdot) \in \mathcal{U}$. Dividing (7.40) by $t_2 - t_1$ and letting $t_2 \to t_1$ it follows that

$$
\mathcal{L}V_{\rm a}(x(t)) \leq r(u(t),y(t)), \qquad t \geq 0, \tag{7.41}
$$

where $x(t)$, $t \geq t_0$, satisfies (7.33) and

$$
\mathcal{L}V_{\rm a}(x(t)) \triangleq V_{\rm a}'(x(t))(f(x(t)) + G(x(t))u(t)) + \frac{1}{2}{\rm tr}\ D^{\rm T}(x(t))V_{\rm a}''(x(t))D(x(t))
$$

denotes the infinitesimal generator of the available storage function along the trajectories $x(t)$, $t \geq t_0$. Now, with $t = t_0$, it follows from (7.41) that

$$
V_{\rm a}'(x_0)(f(x_0)+G(x_0)u) + \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x_0)V_{\rm a}''(x_0)D(x_0) \leq r(u,y(t_0)), \qquad u \in \mathbb{R}^m. \tag{7.42}
$$

Next, let $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be such that

$$
\begin{aligned}
d(x,u) &\triangleq -\mathcal{L}V_{\rm a}(x) + r(u,y) \\
&= -V_{\rm a}'(x)(f(x)+G(x)u) - \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x)V_{\rm a}''(x)D(x) + r(u,h(x)+J(x)u).
\end{aligned} \tag{7.43}
$$

Now, it follows from (7.41) that $d(x,u) \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Furthermore, note that $d(x,u)$ given by (7.43) is quadratic in u, and hence, there exist functions $\ell : \mathbb{R}^n \to \mathbb{R}$ and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that

$$\begin{aligned} d(x,u) &= [\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \\ &= -V_{\mathrm{a}}'(x)(f(x) + G(x)u) - \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{a}}''(x)D(x) + r(u, h(x) + J(x)u) \\ &= -V_{\mathrm{a}}'(x)(f(x) + G(x)u) - \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{a}}''(x)D(x) + (h(x) + J(x)u)^{\mathrm{T}}Q(h(x) + J(x)u) \\ &\quad +2(h(x) + J(x)u)^{\mathrm{T}}Su + u^{\mathrm{T}}Ru. \end{aligned}$$

Now, equating coefficients of equal powers yields (7.35)–(7.37) with $V_{\mathrm{s}}(x) = V_{\mathrm{a}}(x)$ and the positive definiteness of $V_{\mathrm{s}}(x)$, $x \in \mathbb{R}^n$, follows from Theorem 7.2.

Finally, to show (7.39) note that (7.35)–(7.37) can be equivalently written as

$$\begin{bmatrix} \mathcal{A}(x) & \mathcal{B}(x) \\ \mathcal{B}^{\mathrm{T}}(x) & \mathcal{C}(x) \end{bmatrix} = -\begin{bmatrix} \ell^{\mathrm{T}}(x) \\ \mathcal{W}^{\mathrm{T}}(x) \end{bmatrix} \begin{bmatrix} \ell(x) & \mathcal{W}(x) \end{bmatrix} \leq 0, \quad x \in \mathbb{R}^n, \tag{7.44}$$

where $\mathcal{A}(x) \triangleq V_{\mathrm{s}}'(x)f(x) + \frac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x)$, $\mathcal{B}(x) \triangleq \frac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x) + S)$, and $\mathcal{C}(x) \triangleq -(R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x))$. Now, for all invertible $\mathcal{T} \in \mathbb{R}^{(m+1)\times(m+1)}$ (7.44) holds if and only if $\mathcal{T}^{\mathrm{T}}(7.44)\mathcal{T}$ holds. Hence, the equivalence of (7.35)–(7.37) to (7.39) in the case when (7.38) holds follows from the (1,1) block of $\mathcal{T}^{\mathrm{T}}(7.44)\mathcal{T}$, where

$$\mathcal{T} \triangleq \begin{bmatrix} 1 & 0 \\ -\mathcal{C}^{-1}(x)\mathcal{B}^{\mathrm{T}}(x) & I \end{bmatrix}.$$

This completes the proof. □

Note that the assumption of complete stochastic reachability in Theorem 7.5 is needed to establish the existence of a nonnegative-definite storage function $V_{\mathrm{s}}(\cdot)$ while zero-state observability along with the existence of a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $r(\kappa(y), y) < 0$, $y \neq 0$, ensures that $V_{\mathrm{s}}(\cdot)$ is positive definite. In the case where the existence of a two-times continuously differentiable positive-definite storage function $V_{\mathrm{s}}(\cdot)$ is assumed for $\mathcal{G}$, then $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u,y)$ with storage function $V_{\mathrm{s}}(\cdot)$ if and only if (7.35)–(7.37) are satisfied.

Remark 7.2. If (7.35) and (7.39) in Theorem 7.5 are, respectively, replaced by

$$0 = V_{\mathrm{s}}'(x)f(x) + \varepsilon V_{\mathrm{s}}(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{7.45}$$

$$\begin{aligned}0 \geq{}& V_{\mathrm{s}}'(x)f(x) + \varepsilon V_{\mathrm{s}}(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x) \\ &+[\tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x)+S)]\mathcal{N}^{-1}(x)[\tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x)+S)]^{\mathrm{T}},\end{aligned} \tag{7.46}$$

where $\varepsilon > 0$, then it can be shown that Theorem 7.5 provides necessary and sufficient conditions for stochastic exponential dissipativity.

Finally, we provide necessary and sufficient conditions for the case where $\mathcal{G}$ given by (7.33) and (7.34) is lossless with respect to a quadratic supply rate $r(u,y)$.

Theorem 7.6. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be zero-state observable and completely stochastically reachable. $\mathcal{G}$ is stochastically lossless with respect to the quadratic supply rate $r(u,y) = y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru$ if and only if there exists a function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is two-times continuously differentiable and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\mathrm{s}}'(x)f(x) + \tfrac{1}{2}\mathrm{tr}\ D^{\mathrm{T}}(x)V_{\mathrm{s}}''(x)D(x) - h^{\mathrm{T}}(x)Qh(x), \tag{7.47}$$

$$0 = \tfrac{1}{2}V_{\mathrm{s}}'(x)G(x) - h^{\mathrm{T}}(x)(QJ(x)+S), \tag{7.48}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x). \tag{7.49}$$

Proof: The proof is analogous to the proof of Theorem 7.5. □

Using (7.35)–(7.37) it follows that

$$\begin{aligned}\mathbb{E}\left[\int_{t_0}^{t} r(u(s),y(s))\mathrm{d}s|\mathcal{F}_{t_0}\right] ={}& \mathbb{E}\left[V_{\mathrm{s}}(x(t))|\mathcal{F}_{t_0}\right] - V_{\mathrm{s}}(x(t_0)) \\ &+\mathbb{E}\left[\int_{t_0}^{t}[\ell(x(s)) + \mathcal{W}(x(s))u(s)]^{\mathrm{T}}[\ell(x(s)) + \mathcal{W}(x(s))u(s)]\mathrm{d}s|\mathcal{F}_{t_0}\right],\end{aligned} \tag{7.50}$$

which can be interpreted as a *generalized* energy balance equation, where $\mathbb{E}\left[V_{\mathrm{s}}(x(t))|\mathcal{F}_{t_0}\right] - V_{\mathrm{s}}(x(t_0))$ is the stored or accumulated generalized energy of the system and the second path-dependent term on the right corresponds to the dissipated generalized energy of the system.

Rewriting (7.50) as

$$\mathcal{L}V_{\rm s}(x) = r(u,y) - [\ell(x) + \mathcal{W}(x)u]^{\rm T}[\ell(x) + \mathcal{W}(x)u], \tag{7.51}$$

yields a generalized energy conservation equation which shows that the rate of change in generalized system energy, or generalized power, is equal to the external generalized system power input minus the internal generalized system power dissipated.

Note that if $\mathcal{G}$ with a two-times continuously differentiable positive-definite storage function is stochastically dissipative with respect to the quadratic supply rate $r(u,y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$, and if $Q \leq 0$ and $u(t) \stackrel{\rm a.s.}{\equiv} 0$, then it follows that

$$\mathcal{L}V_{\rm s}(x(t)) \stackrel{\rm a.s.}{\leq} y^{\rm T}(t)Qy(t) \stackrel{\rm a.s.}{\leq} 0, \qquad t \geq 0. \tag{7.52}$$

Hence, the zero solution $x(t) \stackrel{\rm a.s.}{\equiv} 0$ of the undisturbed ($u(t) \stackrel{\rm a.s.}{\equiv} 0$) nonlinear stochastic system (7.33) is Lyapunov stable in probability. Alternatively, if $\mathcal{G}$ with a two-times continuously differentiable positive-definite storage function is exponentially dissipative with respect to the quadratic supply rate $r(u,y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$, and if $Q \leq 0$ and $u(t) \stackrel{\rm a.s.}{\equiv} 0$, then it follows that

$$\mathcal{L}V_{\rm s}(x(t)) \stackrel{\rm a.s.}{\leq} -\varepsilon V_{\rm s}(x(t)) + y^{\rm T}(t)Qy(t) \stackrel{\rm a.s.}{\leq} -\varepsilon V_{\rm s}(x(t)), \qquad t \geq 0. \tag{7.53}$$

Hence, the zero solution $x(t) \stackrel{\rm a.s.}{\equiv} 0$ of the undisturbed ($u(t) \stackrel{\rm a.s.}{\equiv} 0$) nonlinear stochastic system (7.33) is asymptotically stable in probability. If, in addition, there exist scalars $\alpha, \beta > 0$ and $p \geq 1$ such that

$$\alpha\|x\|^p \leq V_{\rm s}(x) \leq \beta\|x\|^p, \qquad x \in \mathbb{R}^n, \tag{7.54}$$

then the zero solution $x(t) \stackrel{\rm a.s.}{\equiv} 0$ of the undisturbed ($u(t) \stackrel{\rm a.s.}{\equiv} 0$) nonlinear stochastic dynamical system (7.33) is exponentially p-stable in probability.

Next, we provide several definitions of nonlinear stochastic dynamical systems which are stochastically dissipative or stochastically exponentially dissipative with respect to supply rates of a specific form.

Definition 7.6. A stochastic dynamical system $\mathcal{G}$ of the form (7.1) and (7.2) with $m = l$ is *stochastically passive* (respectively, *stochastically exponentially passive*) if $\mathcal{G}$ is stochastically dissipative (respectively, stochastically exponentially dissipative) with respect to the supply rate $r(u, y) = 2u^{\mathrm{T}}y$.

Definition 7.7. A stochastic dynamical system $\mathcal{G}$ of the form (7.1) and (7.2) is *stochastically nonexpansive* (respectively *stochastically exponentially nonexpansive*) if $\mathcal{G}$ is stochastically dissipative (respectively, stochastically exponentially dissipative) with respect to the supply rate $r(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

Example 7.1. Consider the nonlinear dynamical system given by

$$\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t, \qquad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \qquad t \geq 0, \tag{7.55}$$

$$\mathrm{d}x_2(t) = [-g(x_1(t)) - ax_1(t) + u(t)]\mathrm{d}t + \sigma x_2(t)\mathrm{d}w(t), \qquad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{7.56}$$

$$y(t) = bx_1(t) + x_2(t), \tag{7.57}$$

where $0 < b$, $b + \frac{1}{2}\sigma^2 < a$, $xg(x) > 0$, $x \in \mathbb{R}$, $x \neq 0$, and $g(0) = 0$. To examine the stochastic passivity of (7.55)–(7.57) consider the storage function

$$V_{\mathrm{s}}(x_1, x_2) = \tfrac{\alpha}{2}[\beta a^2 x_1^2 + 2\beta a x_1 x_2 + x_2^2] + \alpha \int_0^{x_1} g(s)\mathrm{d}s, \tag{7.58}$$

where $\alpha > 0$ and $\beta \in (0, 1)$. Note that $V_{\mathrm{s}}(x_1, x_2)$ is positive definite and radially unbounded. Now, computing $\mathcal{L}V_{\mathrm{s}}(x_1, x_2)$ yields

$$\begin{aligned} \mathcal{L}V_{\mathrm{s}}(x_1, x_2) &= \alpha[\beta a^2 x_1 + \beta a x_2 + g(x_1)]x_2 + \alpha(\beta a x_1 + x_2)[-g(x_1) - a x_2 + u] + \tfrac{1}{2}\alpha\sigma^2 x_2^2 \\ &= -\alpha\beta a x_1 g(x_1) + \alpha[(\beta - 1)a + \tfrac{1}{2}\sigma^2]x_2^2 + \alpha(\beta a x_1 + x_2)u. \end{aligned} \tag{7.59}$$

Setting $\alpha = 1$ and $\beta = b/a < 1$ if follows that

$$\mathcal{L}V_{\mathrm{s}}(x_1, x_2) = uy - bx_1 g(x_1) - (a - b - \tfrac{1}{2}\sigma^2)x_2^2 \leq uy, \tag{7.60}$$

which shows that (7.55)–(7.57) is stochastic passive. △

The following results present the nonlinear versions of the Kalman-Yakubovich-Popov *positive real lemma* and the *bounded real lemma* for passive and nonexpansive stochastic dynamical systems.

Corollary 7.2. Let $\mathcal{G}$ be zero-state observable and completely stochastically reachable. $\mathcal{G}$ is stochastically passive if and only if there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\rm s}(\cdot)$ is two-times continuously differentiable and positive definite, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x)f(x) + \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x)V_{\rm s}''(x)D(x) + \ell^{\rm T}(x)\ell(x), \tag{7.61}$$

$$0 = \tfrac{1}{2}V_{\rm s}'(x)G(x) - h^{\rm T}(x) + \ell^{\rm T}(x)\mathcal{W}(x), \tag{7.62}$$

$$0 = J(x) + J^{\rm T}(x) - \mathcal{W}^{\rm T}(x)\mathcal{W}(x). \tag{7.63}$$

If, alternatively,

$$J(x) + J^{\rm T}(x) > 0, \quad x \in \mathbb{R}^n, \tag{7.64}$$

then $\mathcal{G}$ is stochastically passive if and only if there exists a two-times continuously differentiable function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\rm s}(\cdot)$ is positive definite, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} 0 \geq{}& V_{\rm s}'(x)f(x) + \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x)V_{\rm s}''(x)D(x) + [\tfrac{1}{2}V_{\rm s}'(x)G(x) - h^{\rm T}(x)] \\ &\cdot[J(x) + J^{\rm T}(x)]^{-1}[\tfrac{1}{2}V_{\rm s}'(x)G(x) - h^{\rm T}(x)]^{\rm T}. \end{aligned} \tag{7.65}$$

Proof: The result is a direct consequence of Theorem 7.5 with $l = m$, $Q = 0$, $S = I_m$, and $R = 0$. Specifically, with $\kappa(y) = -y$ it follows that $r(\kappa(y), y) = -2y^{\rm T}y < 0$, $y \neq 0$, so that all the assumptions of Theorem 7.5 are satisfied. □

Corollary 7.3. Let $\mathcal{G}$ be zero-state observable and completely stochastically reachable. $\mathcal{G}$ is stochastically nonexpansive if and only if there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\rm s}(\cdot)$ is two-times continuously differentiable and positive definite, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$0 = V_{\rm s}'(x)f(x) + \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x)V_{\rm s}''(x)D(x) + h^{\rm T}(x)h(x) + \ell^{\rm T}(x)\ell(x), \tag{7.66}$$

$$0 = \tfrac{1}{2}V_{\rm s}'(x)G(x) + h^{\rm T}(x)J(x) + \ell^{\rm T}(x)\mathcal{W}(x), \tag{7.67}$$

$$0 = \gamma^2 I_m - J^{\rm T}(x)J(x) - \mathcal{W}^{\rm T}(x)\mathcal{W}(x), \tag{7.68}$$

where $\gamma > 0$. If, alternatively,

$$\gamma^2 I_m - J^{\rm T}(x)J(x) > 0, \quad x \in \mathbb{R}^n, \tag{7.69}$$

then $\mathcal{G}$ is stochastically nonexpansive if and only if there exists a two-times continuously differentiable function $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\rm s}(\cdot)$ is positive definite, $V_{\rm s}(0) = 0$, and, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} 0 \geq\ & V_{\rm s}'(x)f(x) + \tfrac{1}{2}{\rm tr}\ D^{\rm T}(x)V_{\rm s}''(x)D(x) + h^{\rm T}(x)h(x) + [\tfrac{1}{2}V_{\rm s}'(x)G(x) + h^{\rm T}(x)J(x)] \\ & \cdot[\gamma^2 I_m - J^{\rm T}(x)J(x)]^{-1}[\tfrac{1}{2}V_{\rm s}'(x)G(x) + h^{\rm T}(x)J(x)]^{\rm T}. \end{aligned} \tag{7.70}$$

Proof: The result is a direct consequence of Theorem 7.5 with $Q = -I_l$, $S = 0$, and $R = \gamma^2 I_m$. Specifically, with $\kappa(y) = -\frac{1}{2\gamma}y$ it follows that $r(\kappa(y), y) = -\frac{3}{4}y^{\rm T}y < 0$, $y \neq 0$, so that all the assumptions of Theorem 7.5 are satisfied. □

Example 7.2. Consider the nonlinear controlled stochastic dynamical system given by

$$\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t, \quad x_1(0) \overset{\rm a.s.}{=} x_{10}, \quad t \geq 0, \tag{7.71}$$

$$\mathrm{d}x_2(t) = [-a\sin x_1(t) - bx_2(t) + u(t)]\mathrm{d}t + \sigma x_2(t)\mathrm{d}w(t), \quad x_2(0) \overset{\rm a.s.}{=} x_{20}, \tag{7.72}$$

$$y(t) = x_2(t), \tag{7.73}$$

where $a, b > 0$. Note that (7.71)–(7.73) can be written in the state space form (7.33) and (7.34) with $x = [x_1,\ x_2]^{\rm T}$, $f(x) = [x_2,\ -a\sin x_1 - bx_2]^{\rm T}$, $G(x) = [0,\ 1]^{\rm T}$, $D(x) = [0,\ \sigma x_2]^{\rm T}$, $h(x) = x_2$, and $J(x) = 0$. To examine the stochastic nonexpansivity of (7.71)–(7.73) consider the storage function $V_{\rm s}(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$ satisfying $V_{\rm s}(x) \geq 0$, $x \in \mathbb{R}^2$. Now, using Corollary 7.3 it follows from (7.70) that

$$0 \geq \begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -a\sin x_1 - bx_2 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 & \sigma x_2 \end{bmatrix} \begin{bmatrix} a\cos x_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \sigma x_2 \end{bmatrix} + h^2(x)$$
$$+\tfrac{1}{4\gamma^2}\begin{bmatrix} a\sin x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} a\sin x_1 \\ x_2 \end{bmatrix}, \tag{7.74}$$

or, equivalently,

$$0 \geq (1 + \tfrac{1}{2}\sigma^2 - b)h^2(x) + \tfrac{1}{4\gamma^2}h^2(x). \tag{7.75}$$

Hence, (7.75) is satisfied if $\gamma \geq \frac{1}{\sqrt{2[2b-2-\sigma^2]}}$. △

Finally, we note that if (7.61) and (7.65) in Corollary 7.2 are replaced, respectively, by

$$0 = V_{\rm s}'(x)f(x) + \varepsilon V_{\rm s}(x) + \tfrac{1}{2}{\rm tr}\, D^{\rm T}(x)V_{\rm s}''(x)D(x) + \ell^{\rm T}(x)\ell(x), \tag{7.76}$$

$$0 \geq V_{\rm s}'(x)f(x) + \varepsilon V_{\rm s}(x) + \tfrac{1}{2}{\rm tr}\, D^{\rm T}(x)V_{\rm s}''(x)D(x) + [\tfrac{1}{2}V_{\rm s}'(x)G(x) - h^{\rm T}(x)]$$
$$\cdot[J(x) + J^{\rm T}(x)]^{-1}[\tfrac{1}{2}V_{\rm s}'(x)G(x) - h^{\rm T}(x)]^{\rm T}, \tag{7.77}$$

where $\varepsilon > 0$, and (7.66) and (7.70) in Corollary 7.3 are replaced, respectively, by

$$0 = V_{\rm s}'(x)f(x) + \varepsilon V_{\rm s}(x) + \tfrac{1}{2}{\rm tr}\, D^{\rm T}(x)V_{\rm s}''(x)D(x) + h^{\rm T}(x)h(x) + \ell^{\rm T}(x)\ell(x), \tag{7.78}$$

$$0 \geq V_{\rm s}'(x)f(x) + \varepsilon V_{\rm s}(x) + \tfrac{1}{2}{\rm tr}\, D^{\rm T}(x)V_{\rm s}''(x)D(x) + h^{\rm T}(x)h(x) + [\tfrac{1}{2}V_{\rm s}'(x)G(x) + h^{\rm T}(x)J(x)]$$
$$\cdot[\gamma^2 I_m - J^{\rm T}(x)J(x)]^{-1}[\tfrac{1}{2}V_{\rm s}'(x)G(x) + h^{\rm T}(x)J(x)]^{\rm T}, \tag{7.79}$$

where $\varepsilon > 0$ and $\gamma > 0$, then Corollaries 7.2 and 7.3 present the nonlinear versions of the Kalman-Yakubovich-Popov *strict positive real lemma* and *strict bounded real lemma* for exponentially passive and exponentially nonexpansive stochastic dynamical systems, respectively.

7.4. Stability of Feedback Interconnections of Dissipative Stochastic Dynamical Systems

In this section, we consider feedback interconnections of stochastic dissipative dynamical systems. Specifically, using the notion of stochastically dissipative and stochastically exponentially dissipative dynamical systems, with appropriate storage functions and supply rates, we construct Lyapunov functions for interconnected stochastic dynamical systems by appropriately combining storage functions for each subsystem. The feedback system can be nonlinear and either dynamic or static. In the dynamic case, for generality, we allow the nonlinear feedback system (compensator) to be of fixed dimension n_c that may be less than the plant order n.

We begin by considering the nonlinear stochastic dynamical system $\mathcal{G}$ given by (7.33) and (7.34) with the nonlinear stochastic feedback system $\mathcal{G}_c$ given by

$$\mathrm{d}x_c(t) = [f_c(x_c(t)) + G_c(u_c(t), x_c(t))u_c(t)]\mathrm{d}t + D_c(x_c(t))\mathrm{d}w_c(t),\ \ x_c(0) \stackrel{\text{a.s.}}{=} x_{c0},\ \ t \geq 0, \quad (7.80)$$

$$y_c(t) = h_c(u_c(t), x_c(t)) + J_c(u_c(t), x_c(t))u_c(t), \quad (7.81)$$

where $x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^l$, $y_c \in \mathbb{R}^m$, $f_c : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ satisfies $f_c(0) = 0$, $G_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_c \times l}$, $D_c : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c \times d_c}$ satisfies $D_c(0) = 0$, $h_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^m$ satisfies $h_c(0,0) = 0$, $J_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^{m \times l}$, and $w_c(\cdot)$ is a d_c-dimensional independent standard Wiener process such that, for all $t \geq s$, $w_c(t) - w_c(s)$ is independent of $x_c(\tau)$, $w_c(\tau)$, $\tau \leq s$, and $x_c(0)$. We assume that $f_c(\cdot)$, $G_c(\cdot)$, $D_c(\cdot)$, $h_c(\cdot,\cdot)$, and $J_c(\cdot,\cdot)$ are continuously differentiable mappings and the required properties for the existence and uniqueness of solutions in forward time of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ are satisfied. Here and henceforth we assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed, that is, $\det[I_m + J_c(y, x_c)J(x)] \neq 0$ for all y, x, and x_c.

The following results give sufficient conditions for Lyapunov, asymptotic, and exponential mean square stability in probability of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$.

Theorem 7.7. Consider the closed-loop system consisting of the nonlinear stochastic dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ with input-output pairs (u, y) and $(u_c,\ y_c)$, respectively, and with $u_c = y$ and $y_c = -u$. Assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable and stochastically dissipative with respect to the supply rates $r(u, y)$ and $r_c(u_c, y_c)$ and with two-times continuously differentiable, positive definite, and radially unbounded storage functions $V_s(\cdot)$ and

$V_{\rm sc}(\cdot)$, respectively, such that $V_{\rm s}(0) = 0$ and $V_{\rm sc}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $r(u, y) + \sigma r_{\rm c}(u_{\rm c}, y_{\rm c}) \leq 0$. Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is Lyapunov stable in probability.

ii) If $\mathcal{G}_{\rm c}$ is stochastically exponentially dissipative with respect to supply rate $r_{\rm c}(u_{\rm c}, y_{\rm c})$ and $\mathrm{rank}[G_{\rm c}(u_{\rm c}, 0)] = m$, $u_{\rm c} \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally asymptotically stable in probability.

iii) If $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are stochastically exponentially dissipative with respect to supply rates $r(u, y)$ and $r_{\rm c}(u_{\rm c}, y_{\rm c})$, respectively, and $V_{\rm s}(\cdot)$ and $V_{\rm sc}(\cdot)$ are such that there exist constants $\alpha, \alpha_{\rm c}, \beta$, and $\beta_{\rm c} > 0$ such that

$$\alpha\|x\|^2 \leq V_{\rm s}(x) \leq \beta\|x\|^2, \qquad x \in \mathbb{R}^n, \tag{7.82}$$

$$\alpha_{\rm c}\|x_{\rm c}\|^2 \leq V_{\rm sc}(x_{\rm c}) \leq \beta_{\rm c}\|x_{\rm c}\|^2, \qquad x_{\rm c} \in \mathbb{R}^{n_{\rm c}}, \tag{7.83}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally exponentially mean square stable in probability.

Proof: *i*) Consider the Lyapunov function candidate $V(x, x_{\rm c}) = V_{\rm s}(x) + \sigma V_{\rm sc}(x_{\rm c})$. Now, the corresponding infinitesimal generator for the closed-loop system is given by

$$\mathcal{L}V(x, x_{\rm c}) = \mathcal{L}V_{\rm s}(x) + \sigma\mathcal{L}V_{\rm sc}(x_{\rm c}) \leq r(u, y) + \sigma r_{\rm c}(u_{\rm c}, y_{\rm c}) \leq 0, \quad (x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}},$$

which, by Theorem 3.1, implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is Lyapunov stable in probability.

ii) If $\mathcal{G}_{\rm c}$ is stochastically exponentially dissipative it follows that for some scalar $\varepsilon_{\rm c} > 0$,

$$\begin{aligned} \mathcal{L}V(x, x_{\rm c}) &= \mathcal{L}V_{\rm s}(x) + \sigma\mathcal{L}V_{\rm sc}(x_{\rm c}) \\ &\leq -\sigma\varepsilon_{\rm c}V_{\rm sc}(x_{\rm c}) + r(u, y) + \sigma r_{\rm c}(u_{\rm c}, y_{\rm c}) \\ &\leq -\sigma\varepsilon_{\rm c}V_{\rm sc}(x_{\rm c}), \qquad (x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}}. \end{aligned}$$

Since $V_{\rm sc}(x_{\rm c})$ is positive definite, Lyapunov stability in probability of the closed-loop system follows from Theorem 3.1. Moreover, since $V_{\rm sc}(x_{\rm c}) = 0$ only if $x_{\rm c} = 0$, it follows from [75, Cor. 4.2] that $\lim_{t\to\infty} x_{\rm c}(t) \stackrel{\rm a.s.}{=} 0$. Now, since $\operatorname{rank}[G_{\rm c}(u_{\rm c}, 0)] = m$, $u_{\rm c} \in \mathbb{R}^l$, it follows that, $\lim_{t\to\infty} u_{\rm c}(t) = \lim_{t\to\infty} y(t) \stackrel{\rm a.s.}{=} 0$, and hence, by (7.81), $\lim_{t\to\infty} u(t) \stackrel{\rm a.s.}{=} 0$. Now, since $\mathcal{G}$ is zero-state observable it follows that $\lim_{t\to\infty} x(t) \stackrel{\rm a.s.}{=} 0$, and hence, $(x(t), x_{\rm c}(t)) \stackrel{\rm a.s.}{\to} (0, 0)$ as $t \to \infty$. Now, global asymptotic stability in probability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ follows from the fact that $V_{\rm s}(\cdot)$ and $V_{\rm sc}(\cdot)$ are, by assumption, radially unbounded.

iii) Finally, if $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are stochastically exponentially dissipative it follows that

$$
\begin{aligned}
\mathcal{L}V(x, x_{\rm c}) &= \mathcal{L}V_{\rm s}(x) + \sigma\mathcal{L}V_{\rm sc}(x_{\rm c}) \\
&\le -\varepsilon V_{\rm s}(x) - \sigma\varepsilon_{\rm c}V_{\rm sc}(x_{\rm c}) + r(u, y) + \sigma r_{\rm c}(u_{\rm c}, y_{\rm c}) \\
&\le -\varepsilon\alpha\|x\|^2 - \sigma\varepsilon_{\rm c}\alpha_{\rm c}\|x_{\rm c}\|^2 \\
&\le -\min\{\varepsilon\alpha, \sigma\varepsilon_{\rm c}\alpha_{\rm c}\}\|(x, x_{\rm c})\|^2, \qquad (x, x_{\rm c}) \in \mathbb{R}^n \times \mathbb{R}^{n_{\rm c}},
\end{aligned}
$$

and hence, it follows from Theorem 3.1 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ is globally exponentially mean square stable in probability. □

The next result presents Lyapunov, asymptotic, and exponential mean square stability in probability of stochastic dissipative feedback systems with quadratic supply rates.

Theorem 7.8. Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, $Q_{\rm c} \in \mathbb{S}^m$, $S_{\rm c} \in \mathbb{R}^{m\times l}$, and $R_{\rm c} \in \mathbb{S}^l$. Consider the closed-loop system consisting of the nonlinear stochastic dynamical systems $\mathcal{G}$ given by (7.33) and (7.34) and $\mathcal{G}_{\rm c}$ given by (7.80) and (7.81), and assume $\mathcal{G}$ and $\mathcal{G}_{\rm c}$ are zero-state observable. Furthermore, assume $\mathcal{G}$ is stochastically dissipative with respect to the quadratic supply rate $r(u, y) = y^{\rm T}Qy + 2y^{\rm T}Su + u^{\rm T}Ru$ and has a two-times continuously differentiable, positive definite, and radially unbounded storage function $V_{\rm s}(\cdot)$, and $\mathcal{G}_{\rm c}$ is stochastically dissipative with respect to the quadratic supply rate $r_{\rm c}(u_{\rm c}, y_{\rm c}) = y_{\rm c}^{\rm T}Q_{\rm c}y_{\rm c} + 2y_{\rm c}^{\rm T}S_{\rm c}u_{\rm c} + u_{\rm c}^{\rm T}R_{\rm c}u_{\rm c}$ and has a two-times continuously differentiable, positive definite, and radially unbounded storage function $V_{\mathrm{sc}}(\cdot)$. Finally, assume there exists $\sigma > 0$ such that

$$\hat{Q} \triangleq \begin{bmatrix} Q + \sigma R_{\mathrm{c}} & -S + \sigma S_{\mathrm{c}}^{\mathrm{T}} \\ -S^{\mathrm{T}} + \sigma S_{\mathrm{c}} & R + \sigma Q_{\mathrm{c}} \end{bmatrix} \leq 0. \tag{7.84}$$

Then the following statements hold:

i) The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is Lyapunov stable in probability.

ii) If $\mathcal{G}_{\mathrm{c}}$ is stochastically exponentially dissipative with respect to supply rate $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ and $\mathrm{rank}[G_{\mathrm{c}}(u_{\mathrm{c}}, 0)] = m$, $u_{\mathrm{c}} \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally asymptotically stable in probability.

iii) If $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ are stochastically exponentially dissipative with respect to supply rates $r(u, y)$ and $r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}})$ and there exist constants $\alpha, \beta, \alpha_{\mathrm{c}}$, and $\beta_{\mathrm{c}} > 0$ such that (7.82) and (7.83) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally exponentially mean square stable in probability.

iv) If $\hat{Q} < 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally asymptotically stable in probability.

Proof: Statements *i*)–*iii*) are a direct consequence of Theorem 7.7 by noting that

$$r(u, y) + \sigma r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) = \begin{bmatrix} y \\ y_{\mathrm{c}} \end{bmatrix}^{\mathrm{T}} \hat{Q} \begin{bmatrix} y \\ y_{\mathrm{c}} \end{bmatrix},$$

and hence, $r(u, y) + \sigma r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) \leq 0$.

To show *iv*) consider the Lyapunov function candidate $V(x, x_{\mathrm{c}}) = V_{\mathrm{s}}(x) + \sigma V_{\mathrm{sc}}(x_{\mathrm{c}})$. Noting that $u_{\mathrm{c}} = y$ and $y_{\mathrm{c}} = -u$ it follows that the corresponding infinitesimal generator for the closed-loop system is given by

$$\begin{aligned} \mathcal{L}V(x, x_{\mathrm{c}}) &= \mathcal{L}V_{\mathrm{s}}(x) + \sigma \mathcal{L}V_{\mathrm{sc}}(x_{\mathrm{c}}) \\ &\leq r(u, y) + \sigma r_{\mathrm{c}}(u_{\mathrm{c}}, y_{\mathrm{c}}) \\ &= y^{\mathrm{T}} Q y + 2y^{\mathrm{T}} S u + u^{\mathrm{T}} R u + \sigma(y_{\mathrm{c}}^{\mathrm{T}} Q_{\mathrm{c}} y_{\mathrm{c}} + 2y_{\mathrm{c}}^{\mathrm{T}} S_{\mathrm{c}} u_{\mathrm{c}} + u_{\mathrm{c}}^{\mathrm{T}} R_{\mathrm{c}} u_{\mathrm{c}}) \\ &= \begin{bmatrix} y \\ y_{\mathrm{c}} \end{bmatrix}^{\mathrm{T}} \hat{Q} \begin{bmatrix} y \\ y_{\mathrm{c}} \end{bmatrix} \end{aligned}$$

$$\leq 0, \qquad (x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c},$$

which, by Theorem 3.1, implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable in probability. Next, it follows from [75, Cor. 4.1] that $\mathcal{L}V(x, x_c) \stackrel{\text{a.s.}}{\to} 0$ as $t \to \infty$, and note that $\mathcal{L}V(x, x_c) = 0$ if and only if $(y, y_c) = (0, 0)$. Now, since $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable it follows that $(x(t), x_c(t)) \stackrel{\text{a.s.}}{\to} (0, 0)$ as $t \to \infty$. Finally, global asymptotic stability in probability follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded, and hence, $V(x, x_c) \to \infty$ as $\|(x, x_c)\| \to \infty$. □

The following corollary is a direct consequence of Theorem 7.8. For this result note that if a nonlinear stochastic dynamical system $\mathcal{G}$ is stochastic dissipative (respectively, stochastic exponentially dissipative) with respect to a supply rate $r(u, y) = u^{\mathrm{T}}y - \varepsilon u^{\mathrm{T}}u - \hat{\varepsilon}y^{\mathrm{T}}y$, where $\varepsilon, \hat{\varepsilon} \geq 0$, then with $\kappa(y) = ky$, where $k \in \mathbb{R}$ is such that $k(1 - \varepsilon k) < \hat{\varepsilon}$, $r(u, y) = [k(1 - \varepsilon k) - \hat{\varepsilon}]y^{\mathrm{T}}y < 0$, $y \neq 0$. Hence, if $\mathcal{G}$ is zero-state observable it follows from Theorem 7.5 that all storage functions (respectively, exponential storage functions) of $\mathcal{G}$ are positive definite. For the next result, we assume that all storage functions of $\mathcal{G}$ and $\mathcal{G}_c$ are two-times continuously differentiable.

Corollary 7.4. Consider the closed-loop system consisting of the nonlinear stochastic dynamical systems $\mathcal{G}$ given by (7.33) and (7.34) and $\mathcal{G}_c$ given by (7.80) and (7.81), and assume $\mathcal{G}$ and $\mathcal{G}_c$ are zero-state observable. Then the following statements hold:

i) If $\mathcal{G}$ is stochastically passive, $\mathcal{G}_c$ is stochastically exponentially passive, and $\operatorname{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable in probability.

ii) If $\mathcal{G}$ and $\mathcal{G}_c$ are stochastically exponentially passive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (7.82) and (7.83) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is exponentially mean square stable in probability.

iii) If $\mathcal{G}$ is stochastically nonexpansive with gain $\gamma > 0$, $\mathcal{G}_c$ is stochastically exponentially nonexpansive with gain $\gamma_c > 0$, $\text{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, and $\gamma\gamma_c \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable in probability.

iv) If $\mathcal{G}$ and $\mathcal{G}_c$ are stochastically exponentially nonexpansive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (7.82) and (7.83) hold, and with gains $\gamma > 0$ and $\gamma_c > 0$, respectively, such that $\gamma\gamma_c \leq 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is exponentially mean square stable in probability.

Proof: The proof is a direct consequence of Theorem 7.8. Specifically, *i*) and *ii*) follow from Theorem 7.8 with $Q = Q_c = 0$, $S = S_c = I_m$, and $R = R_c = 0$, whereas *iii*) and *iv*) follow from Theorem 7.8 with $Q = -I_l$, $S = 0$, $R = \gamma^2 I_m$, $Q_c = -I_{l_c}$, $S_c = 0$, and $R_c = \gamma_c^2 I_{m_c}$. □

Example 7.3. Consider the controlled damped stochastic Duffing equation given by

$$\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t, \quad x_1(0) \overset{\text{a.s.}}{=} x_{10}, \quad t \geq 0, \tag{7.85}$$

$$\mathrm{d}x_2(t) = [-[2 + x_1^2(t)]x_1(t) - cx_2(t) + u(t)]\mathrm{d}t + \sigma x_2(t)\mathrm{d}w(t), \quad x_2(0) \overset{\text{a.s.}}{=} x_{20}, \tag{7.86}$$

$$y(t) = x_2(t), \tag{7.87}$$

where $c \geq \frac{1}{2}\sigma^2$. Defining $x = [x_1, \ \ x_2]^{\mathrm{T}}$, (7.85)–(7.87) can be written in state space form (7.33) and (7.34) with $f(x) = [x_2, \ \ -(2+x_1^2)x_1 - cx_2]^{\mathrm{T}}$, $G(x) = [0, \ \ 1]^{\mathrm{T}}$, $D(x) = [0, \ \ \sigma x_2]^{\mathrm{T}}$, $h(x) = x_2$, and $J(x) = 0$. With $V_s(x) = x_1^2 + \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, $\ell(x) = \left(\sqrt{c - \frac{1}{2}\sigma^2}\right) x_2$, and $\mathcal{W}(x) \equiv 0$, it follows from Corollary 7.2 that (7.85)–(7.87) is stochastically passive. Now, using Corollary 7.4 we can design a second-order stochastic nonlinear dynamic compensator $\mathcal{G}_c$ to asymptotically stabilize (7.85) and (7.86) in probability. Specifically, it follows from *i*) of Corollary 7.4 that if $\mathcal{G}_c$ given by (7.80) and (7.81) is exponentially passive with $\text{rank}[G_c(u_c, 0)] = 1$, then the negative feedback interconnection of $\mathcal{G}$ given by (7.85)–(7.87) and $\mathcal{G}_c$ is asymptotically stable in probability.

Here, we construct a second-order nonlinear stochastic dynamic compensator $\mathcal{G}_c$ given by

$$\mathrm{d}x_{c1}(t) = [x_{c2}(t) - x_{c1}(t)]\mathrm{d}t, \quad x_{c1}(0) \overset{\text{a.s.}}{=} x_{c10}, \quad t \geq 0, \tag{7.88}$$

$$\mathrm{d}x_{\mathrm{c}2}(t) = [-x_{\mathrm{c}1}^3(t) - x_{\mathrm{c}2}(t) + u_{\mathrm{c}}(t)]\mathrm{d}t + \tfrac{1}{2}x_{\mathrm{c}2}(t)\mathrm{d}w_{\mathrm{c}}(t), \quad x_{\mathrm{c}2}(0) \overset{\text{a.s.}}{=} x_{\mathrm{c}20}, \tag{7.89}$$

$$y_{\mathrm{c}}(t) = x_{\mathrm{c}2}(t). \tag{7.90}$$

Defining $x_{\mathrm{c}} = [x_{\mathrm{c}1}, \quad x_{\mathrm{c}2}]^{\mathrm{T}}$, (7.88)–(7.90) can be written in state space form (7.80) and (7.81) with $f_{\mathrm{c}}(x_{\mathrm{c}}) = [x_{\mathrm{c}2} - x_{\mathrm{c}1}, \ -x_{\mathrm{c}1}^3 - x_{\mathrm{c}2}]^{\mathrm{T}}$, $G_{\mathrm{c}}(x_{\mathrm{c}}) = [0, \quad 1]^{\mathrm{T}}$, $D_{\mathrm{c}}(x_{\mathrm{c}}) = [0, \quad \frac{1}{2}x_{\mathrm{c}2}]^{\mathrm{T}}$, $h_{\mathrm{c}}(x_{\mathrm{c}}) = x_{\mathrm{c}2}$, and $J_{\mathrm{c}}(x_{\mathrm{c}}) \equiv 0$. Note that with $V_{\mathrm{s}}(x_{\mathrm{c}}) = \frac{1}{4}x_{\mathrm{c}1}^4 + \frac{1}{2}x_{\mathrm{c}2}^2$, $\varepsilon \in (0, \frac{7}{8}]$, $\ell(x_{\mathrm{c}}) = [\left(\sqrt{1 - \frac{\varepsilon}{4}}\right) x_{\mathrm{c}1}^2 \quad \left(\sqrt{\frac{7}{8} - \varepsilon}\right) x_{\mathrm{c}2}]^{\mathrm{T}}$, and $\mathcal{W}(x_{\mathrm{c}}) \equiv 0$, it follows from Corollary 7.2 and (7.76)–(7.77) that $\mathcal{G}_{\mathrm{c}}$ is exponentially passive. Hence, Corollary 7.4 guarantees that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_{\mathrm{c}}$ is globally asymptotically stable in probability. Figure 7.1 shows a sample trajectory of the closed-loop system response with initial conditions $[x(0)^{\mathrm{T}}, x_{\mathrm{c}}(0)^{\mathrm{T}}]^{\mathrm{T}} \overset{\text{a.s.}}{=} [1, \ 2, \ 3, \ 4]^{\mathrm{T}}$. △

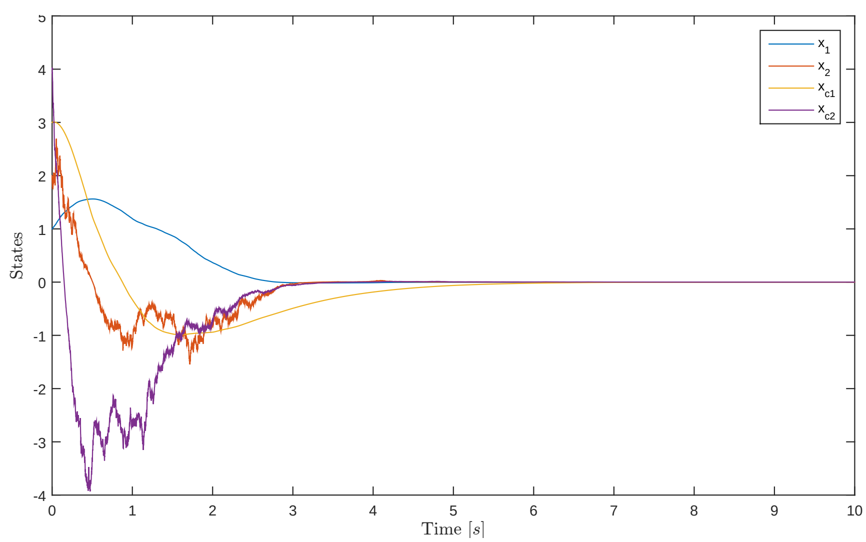

Figure 7.1: Closed-loop system response of feedback interconnection of the system $\mathcal{G}$ and dynamic compensator $\mathcal{G}_c$.

Chapter 8

Summary and Recommendations for Future Research

8.1. Summary

In this dissertation, we presented a unified framework for stability, dissipativity, and optimality for stochastic nonlinear dynamical systems and control. First, in Chapter 2, we developed Lyapunov theorems for semistablility of nonlinear stochastic dynamical systems. In addition, a converse theorem for stochastic semistability was developed using continuous Lyapunov functions. Then, in Chapter 3, an optimal control problem for stochastic stabilization is stated and sufficient conditions are derived to characterize a stochastic optimal nonlinear feedback controller that guarantees asymptotic stability in probability of the closed-loop system. Specifically, we utilized a steady-state stochastic Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function guaranteeing stability in probability of the closed-loop system. This result was then used to develop inverse optimal feedback controllers for affine nonlinear stochastic systems and linear stochastic systems with polynomial and multilinear performance criteria.

In Chapter 4, an optimal control problem for partial-state stochastic stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees asymptotic stability in probability of part of the closed-loop system state.

Specifically, we utilized a steady-state stochastic Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. This result was then used to address optimal linear and nonlinear regulation for linear and nonlinear time-varying stochastic systems with quadratic and nonlinear-nonquadratic performance measures. In addition, we developed inverse optimal feedback controllers for affine nonlinear systems and linear time-varying stochastic systems with polynomial and multilinear performance criteria.

In Chapter 5, an optimal control problem for finite-time, partial-state stochastic stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees finite-time stability in probability of part of the closed-loop system state. Specifically, we utilized a steady-state stochastic Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. This result was then used to develop optimal finite-time stabilizing controllers for nonlinear time-varying stochastic systems. In addition, we developed inverse optimal feedback controllers for affine nonlinear and time-varying stochastic systems.

In Chapter 6, an optimal control strategy for a two-player stochastic differential game problem is stated and sufficient conditions are derived to characterize the stochastic optimal nonlinear feedback control and stopper policies that guarantee asymptotic stability in probability of the closed-loop system. Specifically, we utilized a steady-state stochastic Hamilton-Jacobi-Isaacs framework to characterize optimal nonlinear feedback strategies with a notion of optimality that is directly related to a given Lyapunov function guaranteeing stability in probability of the closed-loop system. This result was then used to develop inverse optimal feedback control and stopper policies for affine nonlinear stochastic differential games and linear stochastic games with polynomial and multilinear performance criteria.

Finally, in Chapter 7, we extended the notion of dissipativity theory for deterministic dynamical systems to controlled Markov diffusion processes and showed the utility of the general concept of dissipation for stochastic systems. Specifically, we provided extended Kalman–Yakubovich–Popov conditions in terms of the drift and diffusion dynamics for characterizing stochastic dissipativity via two-times continuously differentiable storage functions. In addition, using the concepts of stochastic dissipativity for stochastic dynamical systems with appropriate storage functions and supply rates, general stability criteria in probability for feedback interconnections of stochastic dynamical systems were given.

8.2. Recommendations for Future Research

The semistability theorems presented in Chapter 2 require verifying Lyapunov stability for concluding stochastic semistability. However, finding the corresponding Lyapunov function can be a difficult task. To overcome this drawback, we can extend the arc-length-based tests of [20] to stochastic dynamical systems in order to guarantee semistability by testing a condition on the system vector field which avoids proving Lyapunov stability. However, since the sample paths of a stochastic dynamical system may not have an arc-length in the classical sense—due to lack of differentiability of solutions and unbounded variation of sample Wiener paths—stochastic integrals involving nondifferentiable curves as the limiting value of polygonal curves can be used to approximate the arc length of the stochastic system.

Recent technological advances in communications and computation have spurred a broad interest in control of networks and control over networks. Network systems involve distributed decision making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, battle space management, and congestion control in communication networks. A key application area of multiagent network coordination within aerospace systems is cooperative control of unmanned air vehicles for combat, surveillance, and reconnaissance; and swarms of air and space vehicle formations for command and control between heterogeneous air and space vehicles.

In many of the aforementioned applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. Distributed control algorithms can be designed to achieve information consensus that guarantee agreement between agents for a given coordination task. A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in a dynamical network is the existence of a continuum of equilibria representing a state of consensus [53, 54]. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from a state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequently, semistable.

The stochastic semistability framework developed in Chapter 2 can be extended to design consensus controllers for multiagent systems with nonlinear stochastic dynamics under distributed nonlinear consensus protocols. In particular, the results in Chapter 2 can be used as an underpinning for deriving convergence conditions for agreement problems of multiple agents with nonlinear stochastic dynamics over random networks and under nonlinear consensus protocols.

In spite of the appealing nature of the classical stochastic Hamilton-Jacobi-Bellman theory, its current state of development entails limitations in addressing the design of static and dynamic *output-feedback* compensators. In contrast, the simplified and tutorial exposition of the stochastic optimal control framework presented in Chapter 3 can potentially be used to develop a *fixed-structure* stochastic Hamilton-Jacobi-Bellman theory in which one can prespecify the structure of the feedback law with respect to, for example, the order of nonlinearities appearing in the dynamic compensator. The actual gain maps can then be determined by solving algebraic relations in much the same way full-state feedback controllers can be obtained. In this case, the structure of the nonlinear-nonquadratic Lyapunov function, nonlinear-nonquadratic cost functional, and nonlinear feedback controller can be fixed while the performance can be optimized with respect to the controller gains.

To demonstrate how fixed-structure stochastic Hamilton-Jacobi-Bellman synthesis can be performed assume that A (which can denote a closed-loop system) is Hurwitz, let P be given by (3.21), and consider the case where $D(x) = x\sigma^{\mathrm{T}}$ and $L(x), f(x)$, and $V(x)$ are given by (3.24). To satisfy (3.11) we require that (3.26) holds. Equation (3.26) is the basic constraint that must be satisfied by the closed-loop system in order for $J(x_0)$ to be given by (3.12).

Now, for the simplicity of exposition, consider the linear controlled dynamical system with multiplicative noise given by

$$\mathrm{d}x(t) = [Ax(t) + Bu(t)]\,\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{8.1}$$

$$y(t) = Cx(t), \tag{8.2}$$

and constrain the *output feedback* control law to be given by $u = \phi(y)$, where $\phi(\cdot)$ is a finitely parameterized control law (e.g., linear plus cubic plus quintic). Then the closed-loop system takes the form

$$\mathrm{d}x(t) = (Ax(t) + B\phi(Cx(t)))\mathrm{d}t + x(t)\sigma^{\mathrm{T}}\mathrm{d}w(t), \quad x(0) \overset{\text{a.s.}}{=} x_0, \quad t \geq 0, \tag{8.3}$$

which has the form of (2.2) with $f(x)$ given in (3.24). Minimizing $J(x_0)$ given by (3.12) subject to (3.26) now reduces to a system of algebraic relations in the coefficients of the different powers of x. Hence, the proposed framework allows for the synthesis of fixed-structure static and dynamic output-feedback controllers.

Since multiagent network systems can involve information laws governed by nodal dynamics and rerouting strategies that can be modified to minimize waiting times and optimize system throughput, optimality considerations in network systems is of paramount importance. Hence, another key extension to the optimal Hamilton-Jacobi-Bellman theory developed in Chapter 3 is the design of semistabilizing optimal controllers involving controlled dynamical systems with a continuum of equilibria. This will allow for the design of optimal consensus controllers for multiagent networks.

The framework developed in Chapter 3 can also be extended to addressing system robustness to account for changing system parameters. Specifically, a stochastic multiplicative uncertainty model can be used to include modeling of a priori uncertainty in the nonlinear system dynamics. The philosophy of representing uncertain parameters by means of multiplicative white noise is motivated by the Maximum Entropy Principle of Jaynes [59, 60] and statistical analysis [73].

An important extension of the results presented in Chapters 4 is the consideration of optimal partial-state semistabilization. Specifically, optimal partial-state semistabilization as well as finite-time semistabilization is of paramount importance for developing optimal finite-time consensus protocols for addressing finite-time agreement in network systems. Alternatively, in large-scale networks it might be desirable that partial-state synchronization or consensus is sought.

The framework developed in Chapter 5 yields finite interval controllers even though the original cost criterion is defined on the infinite horizon. Hence, one advantage of this approach for certain applications is to obtain finite-interval controllers without the computational complexities of two-point boundary value problems. If the order of the subquadratic state terms appearing in the cost functional is sufficiently small, then the controllers actually optimize a minimum-time cost criterion. Optimal finite-time controllers are usually obtainable using the maximum principle, which generally does not yield feedback controllers. Extensions of the framework developed in Chapter 5 for exploring connections between optimal finite-time stabilization and the classical time-optimal control problem are an important area of research.

The proposed framework can allow us to further explore connections with inverse optimal control, wherein we parametrize a family of finite-time stabilizing sublinear controllers that minimize a derived cost functional involving subquadratic terms. Subquadratic performance criteria have been studied in [50, 93, 94] and have been shown to permit a unified treatment of a broad range of design goals. In addition, as shown in [17] there exist finite-time stable dynamical systems that do not admit smooth Lyapunov functions, and hence, a particularly important extension is the consideration of continuous Lyapunov functions leading to viscosity solutions [31] or, equivalently, a proximal analysis formalism [30], of the resulting stochastic Hamilton-Jacobi-Bellman equations arising in Theorems 5.4 and 5.5.

Finally, the stochastic dissipativity framework developed in Chapter 7 can be extended to explore connections between stochastic dissipativity and stochastic optimal control to address robust stability and robust stabilization problems involving both stochastic and deterministic uncertainty as well as both averaged and worst-case performance criteria. Furthermore, the framework can be used to extend notions from system thermodynamics [47] to develop a stochastic thermodynamic framework for addressing consensus problems for nonlinear multiagent dynamical systems with fixed and switching topologies. Specifically, distributed nonlinear static and dynamic controller architectures for multiagent coordination can be developed that are predicated on system thermodynamic notions resulting in controller architectures involving the exchange of information between agents over random networks that guarantee that the closed-loop dynamical network is consistent with basic stochastic thermodynamic principles. In addition, stochastic dissipativity in the setting of behavioral system [109, 110], where the system storage can be introduced as a *latent* variable associated with a supply rate that is a *manifest* variable, can also be explored.

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